Differential geometry of surfaces
Review
- Gauss map, shape operator
- Second fundamental form
Today
- Curvatures
  - Principal, mean, Gaussian curvature
- Minimal surfaces
- Mean curvature normal
- Isometric, conformal maps
- Gauss theorem

Gauss map $N(p)$

Differential of Gauss map
$$dN_p(v) = N'(0)$$

Second fundamental form
- Defined as the quadratic form $\mathbf{II}_p$ in $T_p(S)$ by
  $$\mathbf{II}_p(v) = -\langle dN_p(v), v \rangle$$

Second fundamental form
- Curvature of normal section at $p$, given by normal $N(p)$ and tangent $v$
Principal curvatures

Properties

• The principal directions are the eigenvectors of the differential of the Gauss map
  \[ dN_p(e_1) = -k_1 e_1, dN_p(e_2) = -k_2 e_2 \]

• The principal directions form an orthonormal basis for \( T_p(S) \)

Questions?

Principal directions

• Direction of maximum curvature

Direction of maximum curvature

[Diweald, Rumpf]

• Which one is minimum/maximum?

Direction of minimum curvature

[Diweald, Rumpf]
Euler formula

- The second fundamental form expressed in the basis $e_1, e_2$
- For a unit vector $v = e_1 \cos \theta + e_2 \sin \theta$

\[ \Pi_p(v) = k_1 \cos^2 \theta + k_2 \sin^2 \theta \]

Gaussian and mean curvature

Gaussian curvature $K$

The determinant of $dN_p$

Mean curvature $H$

Half the negative of the trace of $dN_p$

In terms of principal curvatures

\[ K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2} \]

Mean curvature visualization

A point on the surface is

- **Elliptic** if $\det(dN_p) > 0$, i.e., $k_1 k_2 > 0$
- **Hyperbolic** if $\det(dN_p) < 0$, i.e., $k_1 k_2 < 0$
- **Parabolic** if $\det(dN_p) = 0, dN_p \neq 0$, i.e., $k_1 = 0$ or $k_2 = 0$
- **Planar** if $dN_p = 0$, i.e., $k_1 = k_2 = 0$
- **Umbilical** if $k_1 = k_2$

Questions?

- Express second fundamental form using a local parameterization

\[ x : U \subset \mathbb{R}^2 \rightarrow S \]

\[ x(u, v) = p \in S \]

- Basis for tangent space $x_u, x_v$
  Tangent vectors $x_u u' + x_v v'$
- What is $\Pi_p(u', v'), dN_p(u', v')$?
In local coordinates

- Given curve $\alpha(t) = x(u(t), v(t))$ with $\alpha' = xu' + xv'$ at $p$
- Per definition $dN(\alpha') = N'(u(t), v(t)) = Nu' + Nv'$

In local coordinates

- Note $\langle N, xu \rangle = \langle N, xv \rangle = 0$
- Therefore
  
  $e = -(N_u, xu) = \langle N, x_{uu} \rangle$
  $f = -(N_u, xu) = \langle N, x_{uv} \rangle = -(N_u, x_v)$
  $g = -(N_u, xu) = \langle N, x_{vv} \rangle$

- We don’t need the terms $Nu, Nv$!

Weingarten equations

- With coefficients $E, F, G$ of first fundamental form

  $a_{11} = \frac{1F - eG}{EG - F^2}$
  $a_{12} = \frac{2G - 1F}{EG - F^2}$
  $a_{21} = \frac{eF - fG}{EG - F^2}$
  $a_{22} = \frac{1F - gG}{EG - F^2}$

Weingarten equations

- Because $N_u, N_v \in T_p(S)$ we can write
  
  $N_u = a_{11}x_u + a_{21}x_v$
  $N_v = a_{12}x_u + a_{22}x_v$

  and $dN$ in the basis $x_u, x_v$

  $dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$

- The coefficients $a_{11}, a_{21}, a_{22}$ express the differential of the Gauss map in the basis $x_u, x_v$ of $T_p(S)$

Curvatures

Remember

Curvatures are defined in terms of the differential of the Gauss map

- Gaussian curvature: determinant of $dN$
  
  $K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2}$

- Mean curvature: negative half of trace of $dN$
- Principal curvatures: eigenvalues of $dN$
Outlook

- Theory of smooth surfaces
  - Parameterization
  - Differentiability
- In practice
  - Triangle meshes prevailing

Next time

- Surface representations