Linear Rotation-invariant Coordinates for Meshes
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Introduction
• Interactive mesh editing
  - Intuitive user interface
  - Deformation must be smooth and intuitive
  - Preservation of surface details
• Previous approaches
  - Multiresolution
  - Local frames
Problem—the surface representations are not rotation invariant!

Introduction
Editing with non rotation-invariant coordinates

Introduction
Laplacian editing
Linear rotation-invariant coordinates

Introduction
• A rigid motion-invariant mesh representation
  - Describe surface by local properties
  - Allow interactively editing the mesh while preserving the local surface details

Outline
• Mesh representations
  - Discrete forms
  - Discrete surface equations
• Mesh editing
• Results
• Discussion
Mesh Representation

- Representing the mesh with first and second discrete forms, which are
  - Invariant to rotation and translation
  - Containing enough information to reconstruct the mesh uniquely

Discrete Forms

- First and second discrete forms
  - \( \mathcal{F}(\cdot) : \cup_{i=1}^{N} \Delta_i \rightarrow R \)
  - \( \mathcal{F}^i(\cdot) : \cup_{i=1}^{N} \Delta_i \rightarrow R \)

The First Discrete Form

- Parameterization
  \( \mu = \mu_i x_i + \mu_N x_{N+1} \in \Delta_i \)
  \( \mathcal{F}(\mu) = (\mu, \mu)_{\mathbb{R}^2} \)
  \( = (\mu_i x_i + \mu_N x_{N+1}, \mu_i x_i + \mu_N x_{N+1})_{\mathbb{R}^2} \)
  \( = \mu_i (\tilde{x}_i, N_i)_{\mathbb{R}^2} \)
  \( + \mu_N (\tilde{x}_N, N_N)_{\mathbb{R}^2} \)
  \( O_L := \text{sign} \left( \text{det} (\tilde{x}_i, \tilde{x}_{i+1}, N_i) \right) \)

The Second Discrete Form

- Parameterization
  \( \mu = \mu_i x_i + \mu_N x_{N+1} \in \Delta_i \)
  \( \mathcal{F}(\mu) := \mu_i (\tilde{x}_i, N_i)_{\mathbb{R}^2} + \mu_N (\tilde{x}_{N+1}, N_N)_{\mathbb{R}^2} \)
  \( = \mu_i \tilde{x}_i + \mu_N \tilde{x}_{N+1} \)
  \( \tilde{L}_i = (\tilde{x}_i, N_i)_{\mathbb{R}^2} \)
  Meaning: height function of the 1-ring neighborhood above the tangent planes

Summary of Discrete Forms

- 1st: length
- 2nd: angle
- 3rd: orientation
- 4th: height function
Local Reconstruction

- Given the discrete form coefficients at vertex \( i \), the 1-ring neighborhood of \( i \) is defined up to a rigid transformation

\[
\tilde{x}_i = \tilde{x}_i^N + O(V) \tilde{b}_{i_1} + \bar{L}_i N^i
\]

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Discrete Surface Equations

- Discrete frame
  \((\tilde{x}_i^B, \tilde{x}_i^N, \tilde{N}^i)\)

- Discrete surface equations
  - Encode the difference between adjacent discrete frames

\[
\begin{align*}
\tilde{x}_i^B &= \left(\Gamma_{B_i} + 1\right) \tilde{x}_i^B + \Gamma_{B_i} \tilde{b}_{i_1} + A_{B_i} \tilde{N}^i \\
\tilde{x}_i^N &= \left(\Gamma_{N_i} + 1\right) \tilde{x}_i^N + \Gamma_{N_i} \tilde{b}_{i_1} + A_{N_i} \tilde{N}^i \\
\tilde{N}^i &= \Gamma_{N_i} \tilde{N}^i + \Gamma_{N_i} \tilde{b}_{i_1} + A_{N_i} \tilde{N}^i
\end{align*}
\]

Discrete Surface Equations

- The coefficients in the discrete surface equations are functions of the discrete forms
  - why?
  - The equations express one frame in terms of an adjacent frame
  - Local frame at on vertex has information about neighbors

Global Reconstruction

- The set of equations forms a over-determined sparse linear system

\[
\forall (i, j) \in E
\begin{align*}
\tilde{x}_i - \tilde{x}_j &= \tilde{x}_i^B + \tilde{L}_i N^i = (\tilde{x}_i^B, \tilde{x}_i^N, \tilde{N}^i) \\
\tilde{x}_j &= (\tilde{x}_i^B, \tilde{x}_i^N, \tilde{N}^i)
\end{align*}
\]

\(3|V| \) variables, \( 3|2E| \) equations

- Solve the system in the least square sense!

Global Reconstruction

- Geometry difference equations

\[
\forall (i, j) \in E
\begin{align*}
\tilde{x}_i - \tilde{x}_j &= \tilde{x}_i^B + \tilde{L}_i N^i = (\tilde{x}_i^B, \tilde{x}_i^N, \tilde{N}^i) \\
\tilde{x}_j &= (\tilde{x}_i^B, \tilde{x}_i^N, \tilde{N}^i)
\end{align*}
\]

- Solve the linear system in the least square sense to get the positions of vertices
Put it all together

• Reconstruct the mesh given discrete coefficients, an initial discrete frame and a position of one vertex

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Mesh Editing

• The editing operation is applied by adding linear constraints on the linear systems

Mesh Editing

• Users need to define a ROI and handle when editing the mesh

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More Results

The discrete forms are not scale- or shear-invariant
• Shape interpolation
  - Linear interpolation between discrete forms
  - Two meshes with identical connectivity and different geometries

• Connection with differential geometry
  - First fundamental form
    \[
    (x_0, x_0)_t = \frac{1}{E(u^t)^2 + 2F(v^t) + G(u^t)^2}
    \]
  - Second fundamental form
    \[
    e = -(N_1, x_0) = (N, x_m),
    f = -(N_1, x_0) = (N, x_m) = -(N, x_m),
    g = -(N_1, x_0) = (N, x_m),
    \]

• Thanks for your attention!