CSE291
Topics in Computer Graphics
Mesh Animation

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Today
• Review elastic bodies in 3D
• “Pose space deformation”, Lewis et al.
• Presentation by Iman
• Discussion led by Arash

Stress
• Forces exerted by a piece of material onto its environment
• Measured as force per unit area
• Normal, shear stress
• Stress tensor
• Symmetric!

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \sigma_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \sigma_{zz}
\end{bmatrix}
\]

Stress tensor

Body forces
• External force applied to a piece of material
• Force per volume

\[
\left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \right) = 0
\]

where \( f_x, f_y, f_z \) are body forces (force per volume)

Equilibrium conditions
• Independent of material properties

Deformations and displacements
• Deformation represented by displacement field \( u(x) \)
### Strain
- Measures geometric deformation
- Also called Green tensor, Cauchy's ininitesimal strain tensor

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u}{\partial x} & 0 & 0 \\
0 & \frac{\partial v}{\partial y} & 0 \\
0 & 0 & \frac{\partial w}{\partial z}
\end{bmatrix}
\]

\[
\begin{bmatrix}
u
d\end{bmatrix}
\]

### Stress-strain relationship
- Given geometric deformation of a piece of a body, what are strains (forces)?
- Stress-strain relationship captures physical material properties

![Stress-strain curves for Steel and Aluminum](image)

### Linear elastic bodies
- Generalization of Hooke's law to 3D bodies
- Isotropic materials

\[
d = (1 + \nu)(1 - 2\nu)
\]

### Constitutive equations

#### Strong formulation
- Equilibrium conditions
  \[
  \mathcal{A}(\mathbf{u}) - \begin{bmatrix}
  A_1 \\
  A_2 \\
  A_3
  \end{bmatrix} = \begin{bmatrix}
  \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\
  \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y \\
  \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z
  \end{bmatrix}
  \]

and linear stress-strain relationship
- Linear elastic bodies: linear displacement-strain, linear strain-stress, linear constitutive equations

### FEA recipe
- Derive weak formulation of PDE
  - Multiply PDE with test function
  - Integrate, apply integration by parts
  - Specify constraints on test and trial functions to fulfill boundary conditions
- Galerkin approximation
  - Discretize test and trial functions
  - Plug into weak formulation
  - Derive matrix equation
Weak form

• Weak form, or principle of virtual work

\[ \int_{\Omega} \delta u^T \sigma d\Omega - \int_{\Gamma} \delta u^T t d\Gamma = 0 \]

Internal work \quad External work

• Virtual displacements, test functions \( \delta u \in \mathbb{R}^{6 \times 1} \)
• Virtual strain \( \delta \varepsilon \in \mathbb{R}^{6 \times 1} \), linear function of virtual displacements
• Strain vector \( \sigma \in \mathbb{R}^{6 \times 1} \)

Discretization

Element point of view

• Implementation detail
• Partition domain into elements

\[ \sum_{j=1}^{n} \left( \int_{\Omega_j} \delta \varepsilon^T \sigma d\Omega - \int_{\Gamma_j} \delta \varepsilon^T t d\Gamma \right) = 0 \]

Element point of view

Discretization

Galerkin approximation

Element point of view

• Displacement in each element is

\[ u^{(m)}(x, y, z) = \sum_{i=0}^{k-1} \bar{u}_i^{(m)} N_i^{(m)}(x, y, z) \]

- where \( \bar{u}_i^{(m)} \) \( \in \mathbb{R}^3 \)
- number of shape functions in element \( k \)
- \( (x, y, z) \) lies within element \( m \)

Discretization

Linear shape functions
(global indices)

\[ \mathbf{KU} = \mathbf{R} \]

\[ \sum_{j=1}^{n} a(N_i, N_j) \bar{u}_j = r_i \]

\[ a(N_i, N_j) = \int_{\Omega} (\cdot)(N_i, N_j) \]

Discretization

Linear shape functions
(global indices)

Element point of view

(local indices)

• Maintain map relating element indices and global indices
• Compute entries of stiffness matrix one element after the other (assembly)
• Matrix formulation

\[ \mathbf{KU} = \mathbf{R} \]

- Stiffness matrix \( \mathbf{K} \)
- Force vector \( \mathbf{U} \)
- Unknowns \( \mathbf{R} \)