3 The Geometry of the Gauss Map

3-1. Introduction

As we have seen in Chap. 1, the consideration of the rate of change of the tangent line to a curve $C$ led us to an important geometric entity, namely, the curvature of $C$. In this chapter we shall extend this idea to regular surfaces; that is, we shall try to measure how rapidly a surface $S$ pulls away from the tangent plane $T_p(S)$ in a neighborhood of a point $p \in S$. This is equivalent to measuring the rate of change at $p$ of a unit normal vector field $N$ on a neighborhood of $p$. As we shall see shortly, this rate of change is given by a linear map on $T_p(S)$ which happens to be self-adjoint (see the appendix to Chap. 3). A surprisingly large number of local properties of $S$ at $p$ can be derived from the study of this linear map.

In Sec. 3-2, we shall introduce the relevant definitions (the Gauss map, principal curvatures and principal directions, Gaussian and mean curvatures, etc.) without using local coordinates. In this way, the geometric content of the definitions is clearly brought up. However, for computational as well as for theoretical purposes, it is important to express all concepts in local coordinates. This is taken up in Sec. 3-3.

Sections 3-2 and 3-3 contain most of the material of Chap. 3 that will be used in the remaining parts of this book. The few exceptions will be explicitly pointed out. For completeness, we have proved the main properties of self-adjoint linear maps in the appendix to Chap. 3. Furthermore, for those who have omitted Sec. 2-6, we have included a brief review of orientation for surfaces at the beginning of Sec. 3-2.

Section 3-4 contains a proof of the fact that at each point of a regular surface there exists an orthogonal parametrization, that is, a parametrization such that its coordinate curves meet orthogonally. The techniques used here are interesting in their own right and yield further results. However, for a short course it might be convenient to assume these results and omit the section.

In Sec. 3-5 we shall take up two interesting special cases of surfaces, namely, the ruled surfaces and the minimal surfaces. They are treated independently so that one (or both) of them can be omitted on a first reading.

3-2. The Definition of the Gauss Map and Its Fundamental Properties

We shall begin by briefly reviewing the notion of orientation for surfaces.

As we have seen in Sec. 2-4, given a parametrization $x: U \subset R^3 \rightarrow S$ of a regular surface $S$ at a point $p \in S$, we can choose a unit normal vector at each point of $x(U)$ by the rule

$$N(q) = \frac{x_w \wedge x_v}{|x_w \wedge x_v|}(q), \quad q \in x(U).$$

Thus, we have a differentiable map $N: x(U) \rightarrow R^3$ that associates to each $q \in x(U)$ a unit normal vector $N(q)$.

More generally, if $V \subset S$ is an open set in $S$ and $N: V \rightarrow R^3$ is a differentiable map which associates to each $q \in V$ a unit normal vector at $q$, we say that $N$ is a differentiable field of unit normal vectors on $V$.

It is a striking fact that not all surfaces admit a differentiable field of unit normal vectors defined on the whole surface. For instance, on the Möbius strip of Fig. 3-1 one cannot define such a field. This can be seen intuitively by

![Figure 3-1. The Möbius strip.](image)
going around once along the middle circle of the figure. After one turn, the vector field \( N \) would come back as \(-N\), a contradiction to the continuity of \( N \). Intuitively, one cannot, on the Möbius strip, make a consistent choice of a definite "side"; moving around the surface, we can go continuously to the "other side" without leaving the surface.

We shall say that a regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field \( N \) is called an orientation of \( S \).

For instance, the Möbius strip referred to above is not an orientable surface. Of course, every surface covered by a single coordinate system (for instance, surfaces represented by graphs of differentiable functions) is trivially orientable. Thus, every surface is locally orientable, and orientation is definitely a global property in the sense that it involves the whole surface.

An orientation \( N \) on \( S \) induces an orientation on each tangent space \( T_p(S) \), \( p \in S \), as follows. Define a basis \( \{v, w\} \in T_p(S) \) to be positive if \( \langle v \wedge w, N \rangle \) is positive. It is easily seen that the set of all positive bases of \( T_p(S) \) is an orientation for \( T_p(S) \) (cf. Sec. 1-4).

Further details on the notion of orientation are given in Sec. 2-6. However, for the purpose of Chaps. 3 and 4, the present description will suffice.

Throughout this chapter, \( S \) will denote a regular orientable surface in which an orientation (i.e., a differentiable field of unit normal vectors \( N \)) has been chosen; this will be simply called a surface \( S \) with an orientation \( N \).

**DEFINITION 1.** Let \( S \subset \mathbb{R}^3 \) be a surface with an orientation \( N \). The map \( N: S \rightarrow \mathbb{R}^3 \) takes its values in the unit sphere

\[
S^2 = \{(x, y, z) \in \mathbb{R}^3; \ x^2 + y^2 + z^2 = 1\}
\]

The map \( N: S \rightarrow S^2 \), thus defined, is called the Gauss map of \( S \) (Fig. 3-2).†

It is straightforward to verify that the Gauss map is differentiable. The differential \( dN_p \) of \( N \) at \( p \in S \) is a linear map from \( T_p(S) \) to \( T_{N(p)}(S^2) \). Since \( T_p(S) \) and \( T_{N(p)}(S^2) \) are parallel planes, \( dN_p \) can be looked upon as a linear map on \( T_p(S) \).

The linear map \( dN_p : T_p(S) \rightarrow T_p(S) \) operates as follows. For each parametrized curve \( \alpha(t) \) in \( S \) with \( \alpha(0) = p \), we consider the parametrized curve \( N \alpha(t) = N(\alpha(t)) \) in the sphere \( S^2 \); this amounts to restricting the normal vector \( N \) to the curve \( \alpha(t) \). The tangent vector \( \alpha'(t) = dN_p(\alpha'(0)) \) is a vector in \( T_p(S) \) (Fig. 3-3). It measures the rate of change of the normal vector \( N \), restricted to the curve \( \alpha(t) \), at \( t = 0 \). Thus, \( dN_p \) measures how \( N \) pulls away from \( N(p) \) in a neighborhood of \( p \). In the case of curves, this measure is given by a number, the curvature. In the case of surfaces, this measure is characterized by a linear map.

**Example 1.** For a plane \( P \) given by \( ax + by + cz + d = 0 \), the unit normal vector \( N = (a, b, c)/\sqrt{a^2 + b^2 + c^2} \) is constant, and therefore \( dN = 0 \) (Fig. 3-4).

**Example 2.** Consider the unit sphere

\[
S^2 = \{(x, y, z) \in \mathbb{R}^3; \ x^2 + y^2 + z^2 = 1\}.
\]

If \( \alpha(t) = (x(t), y(t), z(t)) \) is a parametrized curve in \( S^3 \), then

\[
2xx' + 2yy' + 2zz' = 0,
\]

which shows that the vector \((x, y, z)\) is normal to the sphere at the point

†In italics context, letter symbols set in roman rather than italics.
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Figure 3-4. Plane: \( dN_p = 0 \).

\((x, y, z)\). Thus, \( \overline{N} = (x, y, z) \) and \( N = (-x, -y, -z) \) are fields of unit normal vectors in \( S^2 \). We fix an orientation in \( S^2 \) by choosing \( N = (-x, -y, -z) \) as a normal field. Notice that \( N \) points toward the center of the sphere.

Restricted to the curve \( \alpha(t) \), the normal vector

\[
N(t) = (-x(t), -y(t), -z(t))
\]

is a vector function of \( t \), and therefore

\[
dN(x'(t), y'(t), z'(t)) = N'(t) = (-x'(t), -y'(t), -z'(t));
\]

that is, \( dN_p(v) = -v \) for all \( p \in S^2 \) and all \( v \in T_p(S^2) \). Notice that with the choice of \( N \) as a normal field (that is, with the opposite orientation) we would have obtained \( d\overline{N}_p(v) = v \) (Fig. 3-5).

Example 3. Consider the cylinder \( \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = 1\} \). By an argument similar to that of the previous example, we see that \( \overline{N} = (x, y, 0) \)

and \( N = (-x, -y, 0) \) are unit normal vectors at \((x, y, z)\). We fix an orientation by choosing \( N = (-x, -y, 0) \) as the normal vector field.

By considering a curve \((x(t), y(t), z(t))\) contained in the cylinder, that is, with \( x(t)^2 + y(t)^2 = 1 \), we are able to see that, along this curve, \( N(t) = (-x(t), -y(t), 0) \) and therefore

\[
dN(x'(t), y'(t), z'(t)) = N'(t) = (-x'(t), -y'(t), 0).
\]

We conclude the following: If \( v \) is a vector tangent to the cylinder and parallel to the \( z \) axis, then

\[
dN(v) = 0 = 0v;
\]

if \( w \) is a vector tangent to the cylinder and parallel to the \( xy \) plane, then \( dN(w) = -w \) (Fig. 3-6). It follows that the vectors \( v \) and \( w \) are eigenvectors of \( dN \) with eigenvalues 0 and -1, respectively (see the appendix to Chap. 3).

Example 4. Let us analyze the point \( p = (0, 0, 0) \) of the hyperbolic paraboloid \( z = y^2 - x^2 \). For this, we consider a parametrization \( x(u, v) \) given by

\[
x(u, v) = (u, v, u^2 - v^2),
\]

and compute the normal vector \( N(u, v) \). We obtain successively

\[
x_u = (1, 0, -2u),
\]

\[
x_v = (0, 1, 2v),
\]

\[
N = \left( \frac{u}{\sqrt{u^2 + v^2}}, \frac{-v}{\sqrt{u^2 + v^2}}, \frac{1}{2\sqrt{u^2 + v^2}} \right).
\]

Notice that at \( p = (0, 0, 0) \) \( x_u \) and \( x_v \) agree with the unit vectors along the \( x \) and \( y \) axes, respectively. Therefore, the tangent vector at \( p \) to the curve...
\[ z(t) = x(u(t), v(t)) \], with \( \alpha(0) = p \), has, in \( \mathbb{R}^3 \), coordinates \((u'(0), v'(0), 0)\) (Fig. 3-7). Restricting \( N(u, v) \) to this curve and computing \( N'(0) \), we obtain

\[ N'(0) = (2u'(0), -2v'(0), 0), \]

and therefore, at \( p \),

\[ dN_p(u'(0), v'(0), 0) = (2u'(0), -2v'(0), 0). \]

It follows that the vectors \((1, 0, 0)\) and \((0, 1, 0)\) are eigenvectors of \( dN_p \) with eigenvalues \( 2 \) and \(-2\), respectively.

**Example 5.** The method of the previous example, applied to the point \( p = (0, 0, 0) \) of the paraboloid \( z = x^2 + ky^2, k > 0 \), shows that the unit vectors of the \( x \)-axis and the \( y \)-axis are eigenvectors of \( dN_p \), with eigenvalues \( 2 \) and \( 2k \), respectively (assuming that \( N \) is pointing outwards from the region bounded by the paraboloid).

An important fact about \( dN_p \) is contained in the following proposition.

**Proposition 1.** The differential \( dN_p : T_p(S) \rightarrow T_p(S) \) of the Gauss map is a self-adjoint linear map (cf. the appendix to Chap. 3).

**Proof.** Since \( dN_p \) is linear, it suffices to verify that \( \langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle \) for a basis \( \{w_1, w_2\} \) of \( T_p(S) \). Let \( x(u, v) \) be a parametrization of \( S \) at \( p \) and \( \{x_u, x_v\} \) the associated basis of \( T_p(S) \). If \( \alpha(t) = x(u(t), v(t)) \) is a parametrized curve in \( S \), with \( \alpha(0) = p \), we have

\[ dN_p(x'(0)) = dN_p(x_uu'(0) + x_vv'(0)) \]

\[ = \left. \frac{d}{dt} N(u(t), v(t)) \right|_{t=0} \]

\[ = N_uu'(0) + N_vv'(0); \]

in particular, \( dN_p(x_u) = N_u \) and \( dN_p(x_v) = N_v \). Therefore, to prove that \( dN_p \)

is self-adjoint, it suffices to show that

\[ \langle N_u, x_u \rangle = \langle x_u, N_u \rangle. \]

To see this, take the derivatives of \( \langle N, x_u \rangle = 0 \) and \( \langle N, x_v \rangle = 0 \), relative to \( v \) and \( u \), respectively, and obtain

\[ \langle N_u, x_u \rangle + \langle N, x_{uu} \rangle = 0, \]

\[ \langle N_u, x_v \rangle + \langle N, x_{uv} \rangle = 0. \]

Thus,

\[ \langle N_u, x_u \rangle = -\langle N, x_{uu} \rangle = \langle N, x_u \rangle. \]

Q.E.D.

The fact that \( dN_p : T_p(S) \rightarrow T_p(S) \) is a self-adjoint linear map allows us to associate to \( dN_p \) a quadratic form \( Q \) in \( T_p(S) \), given by \( Q(v) = \langle dN_p(v), v \rangle \), \( v \in T_p(S) \) (cf. the appendix to Chap. 3). To obtain a geometric interpretation of this quadratic form, we need a few definitions. For reasons that will be clear shortly, we shall use the quadratic form \( -Q \).

**Definition 2.** The quadratic form \( \Pi_p \), defined in \( T_p(S) \) by \( \Pi_p(v) = -\langle dN_p(v), v \rangle \) is called the second fundamental form of \( S \) at \( p \).

**Definition 3.** Let \( C \) be a regular curve in \( S \) passing through \( p \in S \), \( k \) the curvature of \( C \) at \( p \), and \( \cos \theta = \langle n, N \rangle \), where \( n \) is the normal vector to \( C \) and \( N \) is the normal vector to \( S \) at \( p \). The number \( k_n = k \cos \theta \) is then called the normal curvature of \( C \in S \) at \( p \).

In other words, \( k_n \) is the length of the projection of the vector \( kn \) over the normal to the surface at \( p \), with a sign given by the orientation \( N \) of \( S \) at \( p \) (Fig. 3-8).
Remark. The normal curvature of \( C \) does not depend on the orientation of \( C \) but changes sign with a change of orientation for the surface.

To give an interpretation of the second fundamental form \( \mathcal{I}_p \), consider a regular curve \( C \subset S \) parametrized by \( \alpha(s) \), where \( s \) is the arc length of \( C \), and with \( \alpha(0) = p \). If we denote by \( N(s) \) the restriction of the normal vector \( N \) to the curve \( \alpha(s) \), we have \( \langle N(s), \alpha'(s) \rangle = 0 \). Hence,

\[
\langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle.
\]

Therefore,

\[
\mathcal{I}_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N'(0), \alpha(0) \rangle = \langle N(0), \alpha''(0) \rangle = \langle N, kn \rangle(p) = k_n(p).
\]

In other words, the value of the second fundamental form \( \mathcal{I}_p \) for a unit vector \( v \in T_p(S) \) is equal to the normal curvature of a regular curve passing through \( p \) and tangent to \( v \). In particular, we obtained the following result.

**Proposition 2 (Meusnier).** All curves lying on a surface \( S \) and having at a given point \( p \in S \) the same tangent line have at this point the same normal curvatures.

The above proposition allows us to speak of the normal curvature along a given direction at \( p \). It is convenient to use the following terminology. Given a unit vector \( v \in T_p(S) \), the intersection of \( S \) with the plane containing \( v \) and \( N(p) \) is called the normal section of \( S \) at \( p \) along \( v \) (Fig. 3-9). In a neighborhood of \( p \), a normal section of \( S \) at \( p \) is a regular plane curve on \( S \) whose normal vector \( n \) at \( p \) is \( \pm N(p) \) or zero; its curvature is therefore equal to the absolute value of the normal curvature along \( v \) at \( p \). With this terminology, the above proposition says that the absolute value of the normal curvature at \( p \) of a curve \( \alpha(s) \) is equal to the curvature of the normal section of \( S \) at \( p \) along \( \alpha'(0) \).

**Example 6.** Consider the surface of revolution obtained by rotating the curve \( z = y^4 \) about the \( z \) axis (Fig. 3-10). We shall show that at \( p = (0, 0, 0) \) the differential \( dN_p = 0 \). To see this, we observe that the curvature of the curve \( z = y^4 \) at \( p \) is equal to zero. Moreover, since the \( xy \) plane is a tangent plane to the surface at \( p \), the normal vector \( N(p) \) is parallel to the \( z \) axis. Therefore, any normal section at \( p \) is obtained from the curve \( z = y^4 \) by rotation; hence, it has curvature zero. It follows that all normal curvatures are zero at \( p \), and thus \( dN_p = 0 \).

![Figure 3-10](image-url)

**Example 7.** In the plane of Example 1, all normal sections are straight lines; hence, all normal curvatures are zero. Thus, the second fundamental form is identically zero at all points. This agrees with the fact that \( dN \equiv 0 \).

In the sphere \( S^2 \) of Example 2, with \( N \) as orientation, the normal sections through a point \( p \in S^2 \) are circles with radius 1 (Fig. 3-11). Thus, all normal curvatures are equal to 1, and the second fundamental form is \( \mathcal{I}_p(v) = 1 \) for all \( p \in S^2 \) and all \( v \in T_p(S) \) with \( |v| = 1 \).

![Figure 3-11](image-url)
In the cylinder of Example 3, the normal sections at a point \( p \) vary from a circle perpendicular to the axis of the cylinder to a straight line parallel to the axis of the cylinder, passing through a family of ellipses (Fig. 3-12). Thus, the normal curvatures vary from 1 to 0. It is not hard to see geometrically that 1 is the maximum and 0 is the minimum of the normal curvature at \( p \).

**Figure 3-12.** Normal sections on a cylinder.

However, an application of the theorem on quadratic forms of the appendix to Chap. 3 gives a simple proof of that. In fact, as we have seen in Example 3, the vectors \( w \) and \( v \) (corresponding to the directions of the normal curvatures 1 and 0, respectively) are eigenvectors of \( dN_p \), with eigenvalues \(-1\) and \(0\), respectively. Thus, the second fundamental form takes up its extreme values in these vectors, as we claimed. Notice that this procedure allows us to check that such extreme values are 1 and 0.

We leave it to the reader to analyze the normal sections at the point \( p = (0,0,0) \) of the hyperbolic paraboloid of Example 4.

Let us come back to the linear map \( dN_p \). The theorem of the appendix to Chap. 3 shows that for each \( p \in S \) there exists an orthonormal basis \( \{e_1, e_2\} \) of \( T_p(S) \) such that \( dN_p(e_1) = -k_1 e_1, \) \( dN_p(e_2) = -k_2 e_2 \). Moreover, \( k_1 \) and \( k_2 \) \((k_1 \geq k_2)\) are the maximum and minimum of the second fundamental form \( II_p \), restricted to the unit circle of \( T_p(S) \); that is, they are the extreme values of the normal curvature at \( p \).

**Definition 4.** The maximum normal curvature \( k_1 \) and the minimum normal curvature \( k_2 \) are called the principal curvatures at \( p \); the corresponding directions, that is, the directions given by the eigenvectors \( e_1, e_2 \), are called principal directions at \( p \).

For instance, in the plane all directions at all points are principal directions. The same happens with a sphere. In both cases, this comes from the fact that the second fundamental form at each point, restricted to the unit vectors, is constant (cf. Example 7); thus, all directions are extremals for the normal curvature.

In the cylinder of Example 3, the vectors \( v \) and \( w \) give the principal directions at \( p \), corresponding to the principal curvatures 0 and 1, respectively. In the hyperbolic paraboloid of Example 4, the \( x \) and \( y \) axes are along the principal directions with principal curvatures \(-2\) and \(2\), respectively.

**DEFINITION 5.** If a regular connected curve \( C \) on \( S \) is such that for all \( p \in C \) the tangent line of \( C \) is a principal direction at \( p \), then \( C \) is said to be a line of curvature of \( S \).

**Proposition 3 (Olinde Rodrigues).** A necessary and sufficient condition for a connected regular curve \( C \) on \( S \) to be a line of curvature of \( S \) is that

\[
\frac{d\mathbf{N}(t)}{dt} = \lambda(t)\mathbf{x}'(t),
\]

for any parametrization \( x(t) \) of \( C \), where \( \mathbf{N}(t) = \mathbf{N} \cdot \mathbf{x}(t) \) and \( \lambda(t) \) is a differentiable function of \( t \). In this case, \(-\lambda(t)\) is the (principal) curvature along \( x'(t) \).

**Proof:** It suffices to observe that if \( x'(t) \) is contained in a principal direction, then \( x(t) \) is an eigenvector of \( d\mathbf{N} \) and

\[
d\mathbf{N}(x'(t)) = \mathbf{N}'(t) = \lambda(t)x'(t).
\]

The converse is immediate.

The knowledge of the principal curvatures at \( p \) allows us to compute easily the normal curvature along a given direction of \( T_p(S) \). In fact, let \( v \in T_p(S) \) with \( \|v\| = 1 \). Since \( e_1 \) and \( e_2 \) form an orthonormal basis of \( T_p(S) \), we have

\[
v = e_1 \cos \theta + e_2 \sin \theta,
\]

where \( \theta \) is the angle from \( e_1 \) to \( v \) in the orientation of \( T_p(S) \). The normal curvature \( k_n \) along \( v \) is given by

\[
k_n = II_p(v) = -\langle dN_p(v), v \rangle
\]

\[
= -\langle dN_p(e_1 \cos \theta + e_2 \sin \theta), e_1 \cos \theta + e_2 \sin \theta \rangle
\]

\[
= \langle e_1, e_2 \rangle \cos \theta + e_2 k_2 \sin \theta, e_1 \cos \theta + e_2 \sin \theta \rangle
\]

\[
= k_1 \cos^2 \theta + k_2 \sin^2 \theta.
\]

The last expression is known classically as the Euler formula; actually, it is just the expression of the second fundamental form in the basis \( \{e_1, e_2\} \).
Given a linear map $A: V \rightarrow V$ of a vector space of dimension 2 and given a basis $\{v_1, v_2\}$ of $V$, we recall that
\[
\text{determinant of } A = a_{11}a_{22} - a_{12}a_{21}, \quad \text{trace of } A = a_{11} + a_{22},
\]
where $(a_{ij})$ is the matrix of $A$ in the basis $\{v_1, v_2\}$. It is known that these numbers do not depend on the choice of the basis $\{v_1, v_2\}$ and are, therefore, attached to the linear map $A$.

In our case, the determinant of $dN$ is the product $(-k_1)(-k_2) = k_1k_2$ of the principal curvatures, and the trace of $dN$ is the negative $-(k_1 + k_2)$ of the sum of principal curvatures. If we change the orientation of the surface, the determinant does not change (the fact that the dimension is even is essential here); the trace, however, changes sign.

**Definition 6.** Let $p \in S$ and let $dN_p: T_p(S) \rightarrow T_p(S)$ be the differential of the Gauss map. The determinant of $dN_p$ is the Gaussian curvature $K$ of $S$ at $p$. The negative of half of the trace of $dN_p$ is called the mean curvature $H$ of $S$ at $p$.

In terms of the principal curvatures we can write

\[
K = k_1k_2, \quad H = \frac{k_1 + k_2}{2}.
\]

**Definition 7.** A point of a surface $S$ is called

1. Elliptic if $\det(dN_p) > 0$.
2. Hyperbolic if $\det(dN_p) < 0$.
3. Parabolic if $\det(dN_p) = 0$, with $dN_p \neq 0$.
4. Planar if $\det dN_p = 0$.

It is clear that this classification does not depend on the choice of the orientation.

At an elliptic point the Gaussian curvature is positive. Both principal curvatures have the same sign, and therefore all curves passing through this point have their normal vectors pointing toward the same side of the tangent plane. The points of a sphere are elliptic points. The point $(0, 0, 0)$ of the paraboloid $z = x^2 + ky^2$, $k > 0$ (cf. Example 5), is also an elliptic point.

At a hyperbolic point, the Gaussian curvature is negative. The principal curvatures have opposite signs, and therefore there are curves through $p$ whose normal vectors at $p$ point toward any of the sides of the tangent plane at $p$. The point $(0, 0, 0)$ of the hyperbolic paraboloid $z = y^2 - x^2$ (cf. Example 4) is a hyperbolic point.

At a parabolic point, the Gaussian curvature is zero, but one of the principal curvatures is not zero. The points of a cylinder (cf. Example 3) are parabolic points.

Finally, at a planar point, all principal curvatures are zero. The points of a plane trivially satisfy this condition. A nontrivial example of a planar point was given in Example 6.

**Definition 8.** If at $p \in S$, $k_1 = k_2$, then $p$ is called an umbilical point of $S$; in particular, the planar points $(k_1 = k_2 = 0)$ are umbilical points.

All the points of a sphere and a plane are umbilical points. Using the method of Example 6, we can verify that the point $(0, 0, 0)$ of the paraboloid $z = x^2 + y^2$ is a (nonplanar) umbilical point.

We shall now prove the interesting fact that the only surfaces made up entirely of umbilical points are essentially spheres and planes.

**Proposition 4.** If all points of a connected surface $S$ are umbilical points, then $S$ is either contained in a sphere or in a plane.

**Proof.** Let $p \in S$ and let $x(u, v)$ be a parametrization of $S$ at $p$ such that the coordinate neighborhood $V$ is connected.

Since each $q \in V$ is an umbilical point, we have, for any vector $w = a_1 x_u + a_2 x_v$ in $T_q(S)$,

\[
dN(w) = \lambda(q)w,
\]

where $\lambda = \lambda(q)$ is a real differentiable function in $V$.

We first show that $\lambda(q)$ is constant in $V$. For that, we write the above equation as

\[
N_u a_1 + N_v a_2 = \lambda(x_u a_1 + x_v a_2);
\]

hence, since $w$ is arbitrary,

\[
N_u = \lambda x_u, \quad N_v = \lambda x_v.
\]

Differentiating the first equation in $v$ and the second one in $u$ and subtracting the resulting equations, we obtain

\[
\lambda_u x_v - \lambda_v x_u = 0.
\]

Since $x_v$ and $x_u$ are linearly independent, we conclude that

\[
\lambda_u = \lambda_v = 0,
\]

for all $q \in V$. Since $V$ is connected, $\lambda$ is constant in $V$, as we claimed.
If \( \lambda \equiv 0 \), \( N_x = N_y = 0 \) and therefore \( N = N_0 \) = constant in \( V \). Thus,
\[
\langle x(u, v), N_0 \rangle_u = \langle x(u, v), N_0 \rangle_v = 0;
\]
hence,
\[
\langle x(u, v), N_0 \rangle = \text{const.,}
\]
and all points \( x(u, v) \) of \( V \) belong to a plane.

If \( \lambda \neq 0 \), then the point \( x(u, v) - (1/\lambda)N(u, v) = y(u, v) \) is fixed, because
\[
(x(u, v) - \frac{1}{\lambda}N(u, v))_u = (x(u, v) - \frac{1}{\lambda}N(u, v))_v = 0.
\]
Since
\[
| x(u, v) - y |^2 = \frac{1}{\lambda^2},
\]
all points of \( V \) are contained in a sphere of center \( y \) and radius \( 1/|\lambda| \).

This proves the proposition locally, that is, for a neighborhood of a point \( p \in S \). To complete the proof we observe that, since \( S \) is connected, given any other point \( r \in S \), there exists a continuous curve \( \alpha: [0, 1] \to S \) with \( \alpha(0) = p \), \( \alpha(1) = r \). For each point \( \alpha(t) \in S \) of this curve there exists a neighborhood \( V_t \) in \( S \) contained in a sphere or in a plane and such that \( \alpha^{-1}(V_t) \) is an open interval of \([0, 1]\). The union \( \bigcup \alpha^{-1}(V_t), t \in [0, 1], \) covers \([0, 1]\) and since \([0, 1]\) is a closed interval, it is covered by finitely many elements of the family \( \{\alpha^{-1}(V_t)\} \) (cf. the Heine-Borel theorem, Prop. 6 of the appendix to Chap. 2). Thus, \( \alpha([0, 1]) \) is covered by a finite number of the neighborhoods \( V_t \).

If the points of one of these neighborhoods are on a plane, all the others will be on the same plane. Since \( r \) is arbitrary, all the points of \( S \) belong to this plane.

If the points of one of these neighborhoods are on a sphere, the same argument shows that all points on \( S \) belong to a sphere, and this completes the proof.

Q.E.D.

**DEFINITION 9.** Let \( p \) be a point in \( S \). An asymptotic direction of \( S \) at \( p \) is a direction of \( T_p(S) \) for which the normal curvature is zero. An asymptotic curve of \( S \) is a regular connected curve \( C \subset S \) such that for each \( p \in C \) the tangent line of \( C \) at \( p \) is an asymptotic direction.

It follows at once from the definition that at an elliptic point there are no asymptotic directions.

A useful geometric interpretation of the asymptotic directions is given by means of the Dupin indicatrix, which we shall now describe.

Let \( p \) be a point in \( S \). The Dupin indicatrix at \( p \) is the set of vectors \( w \) of \( T_p(S) \) such that \( H_p(w) = \pm 1 \).

To write the equations of the Dupin indicatrix in a more convenient form, let \((\xi, \eta)\) be the cartesian coordinates of \( T_p(S) \) in the orthonormal basis \( \{e_1, e_2\} \), where \( e_1 \) and \( e_2 \) are eigenvectors of \( dN_p \). Given \( w \in T_p(S) \), let \( \rho \) and \( \theta \) be "polar coordinates" defined by \( w = \rho v \), with \(|v| = 1\) and \( v = e_1 \cos \theta + e_2 \sin \theta \), if \( \rho \neq 0 \). By Euler's formula,
\[
\pm 1 = H_p(w) = \rho^2 H_p(v) = k_1 \rho^2 \cos^2 \theta + k_2 \rho^2 \sin^2 \theta = k_1 \xi^2 + k_2 \eta^2,
\]
where \( w = \xi e_1 + \eta e_2 \). Thus, the coordinates \((\xi, \eta)\) of a point of the Dupin indicatrix satisfy the equation
\[
k_1 \xi^2 + k_2 \eta^2 = \pm 1;
\]
hence, the Dupin indicatrix is a union of conics in \( T_p(S) \). We notice that the normal curvature along the direction determined by \( w \) is \( k_1(v) = H_p(v) = \pm 1/|\rho|^2 \).

For an elliptic point, the Dupin indicatrix is an ellipse (\( k_1 \) and \( k_2 \) have the same sign); this ellipse degenerates into a circle if the point is an umbilical nonplanar point (\( k_1 = k_2 \neq 0 \)).

For a hyperbolic point, \( k_1 \) and \( k_2 \) have opposite signs. The Dupin indicatrix is therefore made up of two hyperbolas with a common pair of asymptotic lines (Fig. 3-13). Along the directions of the asymptotes, the normal curvature is zero; they are therefore asymptotic directions. This justifies the terminology and shows that a hyperbolic point has exactly two asymptotic directions.

![Figure 3-13. The Dupin indicatrix.](image)

For a parabolic point, one of the principal curvatures is zero, and the Dupin indicatrix degenerates into a pair of parallel lines. The common direction of these lines is the only asymptotic direction at the given point.

In Example 5 of Sec. 3-3 we shall show an interesting property of the Dupin indicatrix.

Closely related with the concept of asymptotic direction is the concept of conjugate directions, which we shall now define.
DEFINITION 10. Let \( p \) be a point on a surface \( S \). Two nonzero vectors \( w_1, w_2 \in T_p(S) \) are conjugate if \( \langle dN_p(w_1), w_2 \rangle = \langle w_1, dN_p(w_2) \rangle = 0 \). Two directions \( r_1, r_2 \) at \( p \) are conjugate if a pair of nonzero vectors \( w_1, w_2 \) parallel to \( r_1 \) and \( r_2 \), respectively, are conjugate.

It is immediate to check that the definition of conjugate directions does not depend on the choice of the vectors \( w_1 \) and \( w_2 \) on \( r_1 \) and \( r_2 \).

It follows from the definition that the principal directions are conjugate and that an asymptotic direction is conjugate to itself. Furthermore, at a nonplanar umbilic, every orthogonal pair of directions is a pair of conjugate directions, and at a planar umbilic each direction is conjugate to any other direction.

Let us assume that \( p \in S \) is not an umbilical point, and let \( \{e_1, e_2\} \) be the orthonormal basis of \( T_p(S) \) determined by \( dN_p(e_1) = -k_1 e_1, \ dN_p(e_2) = -k_2 e_2 \). Let \( \theta \) and \( \varphi \) be the angles that a pair of directions \( r_1 \) and \( r_2 \) make with \( e_1 \). We claim that \( r_1 \) and \( r_2 \) are conjugate if and only if

\[
    k_1 \cos \theta \cos \varphi = -k_2 \sin \theta \sin \varphi.
\]

In fact, \( r_1 \) and \( r_2 \) are conjugate if and only if the vectors

\[
    w_1 = e_1 \cos \theta + e_2 \sin \theta, \quad w_2 = e_1 \cos \varphi + e_2 \sin \varphi
\]

are conjugate. Thus,

\[
    0 = \langle dN_p(w_1), w_2 \rangle = -k_1 \cos \theta \cos \varphi - k_2 \sin \theta \sin \varphi.
\]

Hence, condition (2) follows.

When both \( k_1 \) and \( k_2 \) are nonzero (i.e., \( p \) is either an elliptic or a hyperbolic point), condition (2) leads to a geometric construction of conjugate directions.

![Figure 3-14. Construction of conjugate directions.](image)

**Definition of the Gauss Map**

In terms of the Dupin indicatrix at \( p \). We shall describe the construction at an elliptic point, the situation at a hyperbolic point being similar. Let \( r \) be a straight line through the origin of \( T_p(S) \) and consider the intersection points \( q_1, q_2 \) of \( r \) with the Dupin indicatrix (Fig. 3-14). The tangent lines of the Dupin indicatrix at \( q_1 \) and \( q_2 \) are parallel, and their common direction \( r' \) is conjugate to \( r \). We shall leave the proofs of these assertions to the Exercises (Exercise 12).

**EXERCISES**

1. Show that at a hyperbolic point, the principal directions bisect the asymptotic directions.

2. Show that if a surface is tangent to a plane along a curve, then the points of this curve are either parabolic or planar.

3. Let \( C \subset S \) be a regular curve on a surface \( S \) with Gaussian curvature \( K > 0 \). Show that the curvature \( k \) of \( C \) at \( p \) satisfies

\[
    k \geq \min(|k_1|, |k_2|),
\]

where \( k_1 \) and \( k_2 \) are the principal curvatures of \( S \) at \( p \).

4. Assume that a surface \( S \) has the property that \( |k_1| \leq 1, |k_2| \leq 1 \) everywhere. Is it true that the curvature \( k \) of a curve on \( S \) also satisfies \( |k| \leq 1 \)?

5. Show that the mean curvature \( H \) at \( p \in S \) is given by

\[
    H = \frac{1}{\pi} \int_0^\pi k_n(\theta) \, d\theta,
\]

where \( k_n(\theta) \) is the normal curvature at \( p \) along a direction making an angle \( \theta \) with a fixed direction.

6. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point \( p \in S \), is constant.

7. Show that if the mean curvature is zero at a nonplanar point, then this point has two orthogonal asymptotic directions.

8. Describe the region of the unit sphere covered by the image of the Gauss map of the following surfaces:
   a. Paraboloid of revolution \( z = x^2 + y^2 \).
   b. Hyperboloid of revolution \( x^2 + y^2 - z^2 = 1 \).
   c. Catenoid \( x^2 + y^2 = \cosh z \).

9. Prove that
   a. The image \( N \circ \alpha \) by the Gauss map \( N: S \rightarrow S^2 \) of a parametrized regular
The Geometry of the Gauss Map

curve \( \alpha : I \rightarrow S \) which contains no planar or parabolic points is a parametrized regular curve on the sphere \( S^2 \) (called the spherical image of \( \alpha \)).

b. If \( C = \alpha(t) \) is a line of curvature, and \( k \) is its curvature at \( p \), then

\[
k = |k_p k_N|,
\]

where \( k_p \) is the normal curvature at \( p \) along the tangent line of \( C \) and \( k_N \) is the curvature of the spherical image \( N(C) \subset S^2 \) at \( N(p) \).

10. Assume that the osculating plane of a line of curvature \( C \subset S \), which is nowhere tangent to an asymptotic direction, makes a constant angle with the tangent plane of \( S \) along \( C \). Prove that \( C \) is a plane curve.

11. Let \( p \) be an elliptic point of a surface \( S \), and let \( r \) and \( r' \) be conjugate directions at \( p \). Let \( r \) vary in \( T_p(S) \) and show that the minimum of the angle \( \alpha \) with \( r' \) is reached at a unique pair of directions in \( T_p(S) \) that are symmetric with respect to the principal directions.

12. Let \( p \) be a hyperbolic point of a surface \( S \), and let \( r \) be a direction in \( T_p(S) \). Describe and justify a geometric construction to find the conjugate direction \( r' \) of \( r \) in terms of the Dupin indicatrix (cf. the construction at the end of Sec. 3-2).

*13. (Theorem of Beltrami-Enneper.) Prove that the absolute value of the torsion \( \tau \) at a point of an asymptotic curve, whose curvature is nowhere zero, is given by

\[
|\tau| = \sqrt{-K},
\]

where \( K \) is the Gaussian curvature of the surface at the given point.

*14. If the surface \( S_1 \) intersects the surface \( S_2 \) along the regular curve \( C \), then the curvature \( k \) of \( C \) at \( p \in C \) is given by

\[
k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 + 2\lambda_1 \lambda_2 \cos \theta,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the normal curvatures at \( p \), along the tangent line to \( C \), of \( S_1 \) and \( S_2 \), respectively, and \( \theta \) is the angle made up by the normal vectors of \( S_1 \) and \( S_2 \) at \( p \).

15. (Theorem of Joachimsthal.) Suppose that \( S_1 \) and \( S_2 \) intersect along a regular curve \( C \) and make an angle \( \theta(p) \), \( p \in C \). Assume that \( C \) is a line of curvature of \( S_1 \). Prove that \( \theta(p) \) is constant if and only if \( C \) is a line of curvature of \( S_2 \).

*16. Show that the meridians of a torus are lines of curvature.

17. Show that if \( H = 0 \) on \( S \) and \( S \) has no planar points, then the Gauss map \( N : S \rightarrow S^2 \) has the following property:

\[
\langle dN_p(w_1), dN_p(w_2) \rangle = -K(p)\langle w_1, w_2 \rangle
\]

for all \( p \in S \) and all \( w_1, w_2 \in T_p(S) \). Show that the above condition implies that the angle of two intersecting curves on \( S^2 \) and the angle of their spherical images (cf. Exercise 9) are equal up to a sign.

The Gauss Map in Local Coordinates

*18. Let \( \lambda_1, \ldots, \lambda_m \) be the normal curvatures at \( p \in S \) along directions making angles \( 0, 2\pi/m, \ldots, (m-1)2\pi/m \) with a principal direction \( m > 2 \). Prove that

\[
\lambda_1 + \cdots + \lambda_m = mh,
\]

where \( H \) is the mean curvature at \( p \).

*19. Let \( C \subset S \) be a regular curve in \( S \). Let \( p \in C \) and \( \alpha(s) \) be a parametrization of \( C \) in \( p \) by arc length so that \( \alpha(0) = p \). Choose in \( T_p(S) \) an orthonormal positive basis \( \{t, h\} \), where \( t = \alpha'(0) \). The geodesic torsion \( \tau_\alpha \) of \( C \subset S \) at \( p \) is defined by

\[
\tau_\alpha = \frac{dN}{ds}(0, h).
\]

Prove that

a. \( \tau_\alpha = (k_1 - k_2) \cos \phi \sin \phi \), where \( \phi \) is the angle from \( e_1 \) to \( t \).

b. If \( \tau \) is the torsion of \( C \), \( n \) is the (principal) normal vector of \( C \) and \( \cos \theta = \langle N, n \rangle \), then

\[
\frac{d\theta}{ds} = \tau - \tau_\alpha.
\]

c. The lines of curvature of \( S \) are characterized by having geodesic torsion identically zero.

*20. (Dupin's Theorem.) Three families of surfaces are said to form a triply orthogonal system in an open set \( U \subset \mathbb{R}^3 \) if a unique surface of each family passes through each point \( p \in U \) and if the three surfaces that pass through \( p \) are pairwise orthogonal. Use part c of Exercise 19 to prove Dupin's theorem: The surfaces of a triply orthogonal system intersect each other in lines of curvature.

3-3. The Gauss Map in Local Coordinates

In the preceding section, we introduced some concepts related to the local behavior of the Gauss map. To emphasize the geometry of the situation, the definitions were given without the use of a coordinate system. Some simple examples were then computed directly from the definitions; this procedure, however, is inefficient in handling general situations. In this section, we shall obtain the expressions of the second fundamental form and of the differential of the Gauss map in a coordinate system. This will give us a systematic method for computing specific examples. Moreover, the general expressions thus obtained are essential for a more detailed investigation of the concepts introduced above.

All parametrization \( x : U \subset \mathbb{R}^2 \rightarrow S \) considered in this section are assumed to be compatible with the orientation \( N \) of \( S \); that is, in \( x(U) \),
\[ N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}. \]

Let \( \mathbf{x}(u, v) \) be a parametrization at a point \( p \in S \) of a surface \( S \), and let \( \mathbf{a}(t) = \mathbf{x}(u(t), v(t)) \) be a parametrized curve on \( S \), with \( \mathbf{a}(0) = p \). To simplify the notation, we shall make the convention that all functions to appear below denote their values at the point \( p \).

The tangent vector to \( \mathbf{a}(t) \) at \( p \) is \( \mathbf{a}' = \mathbf{x}_u u' + \mathbf{x}_v v' \) and

\[ dN(\mathbf{a}') = N'(u(t), v(t)) = N_u u' + N_v v'. \]

Since \( N_u \) and \( N_v \) belong to \( T_p(S) \), we may write

\[ N_u = a_{11}x_u + a_{12}x_v, \]
\[ N_v = a_{13}x_u + a_{22}x_v, \]

and therefore,

\[ dN(\mathbf{a}') = (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v; \]

hence,

\[ dN(\begin{pmatrix} u' \\ v' \end{pmatrix}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \]

This shows that in the basis \( \{x_u, x_v\} \), \( dN \) is given by the matrix \( (a_{ij}) \), \( i, j = 1, 2 \). Notice that this matrix is not necessarily symmetric, unless \( \{x_u, x_v\} \) is an orthonormal basis.

On the other hand, the expression of the second fundamental form in the basis \( \{x_u, x_v\} \) is given by

\[ II(\mathbf{a}') = -\langle dN(\mathbf{a}'), \mathbf{a}'' \rangle = -\langle N_u u'' + N_v v'', x_u u' + x_v v' \rangle 
= e(u')^2 + 2f u' v' + g(v')^2, \]

where, since \( \langle N, x_u \rangle = \langle N, x_v \rangle = 0 \),

\[ e = -\langle N_u, x_u \rangle = \langle N, x_u \rangle, \]
\[ f = -\langle N_v, x_u \rangle = \langle N, x_u \rangle = \langle N, x_v \rangle = -\langle N_v, x_v \rangle, \]
\[ g = -\langle N_v, x_v \rangle = \langle N, x_v \rangle. \]

We shall now obtain the values of \( a_{ij} \) in terms of the coefficients \( e, f, g \).

From Eq. (1), we have

\[ -f = \langle N_u, x_v \rangle = a_{11}F + a_{22}G, \]
\[ -f = \langle N_v, x_u \rangle = a_{12}E + a_{22}F, \]
\[ -e = \langle N_u, x_u \rangle = a_{11}E + a_{21}F, \]
\[ -g = \langle N_v, x_v \rangle = a_{12}F + a_{22}G, \]

where \( E, F, \) and \( G \) are the coefficients of the first fundamental form in the basis \( \{x_u, x_v\} \) (cf. Sec. 2-5). Relations (2) may be expressed in matrix form by

\[ \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}; \]

hence,

\[ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}, \]

where \( (\cdot)^{-1} \) means the inverse of \( (\cdot) \). It is easily checked that

\[ \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}, \]

whence the following expressions for the coefficients \( a_{ij} \) of the matrix of \( dN \) in the basis \( \{x_u, x_v\} \):

\[ a_{11} = \frac{fE - gF}{EG - F^2}, \]
\[ a_{12} = \frac{gE - fF}{EG - F^2}, \]
\[ a_{21} = \frac{eF - gE}{EG - F^2}, \]
\[ a_{22} = \frac{fE - gF}{EG - F^2}. \]

For completeness, it should be mentioned that relations (1), with the above values, are known as the equations of Weingarten.

From Eq. (3) we immediately obtain

\[ K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2}. \]

To compute the mean curvature, we recall that \( -k_1, -k_2 \) are the eigenvalues of \( dN \). Therefore, \( k_1 \) and \( k_2 \) satisfy the equation

\[ dN(v) = -kv = -kTv \quad \text{for some} \; v \in T_p(S), \; v \neq 0, \]

where \( I \) is the identity map. It follows that the linear map \( dN + kI \) is not invertible; hence, it has zero determinant. Thus,

\[ \det(a_{11} + k, a_{12} \\ a_{21}, a_{22} + k) = 0 \]

or

\[ k^2 + k(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0. \]
Since \( k_1 \) and \( k_2 \) are the roots of the above quadratic equation, we conclude that

\[
H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2} \frac{eG - 2fF + gE}{\sqrt{EG - F^2}},
\]  

(5)
hence,

\[
k^2 - 2HK + K = 0,
\]

and therefore,

\[
k = H \pm \sqrt{H^2 - K}.
\]

(6)

From this relation, it follows that if we choose \( k_1(q) \geq k_2(q) \), \( q \in S \), then the functions \( k_1 \) and \( k_2 \) are continuous in \( S \). Moreover, \( k_1 \) and \( k_2 \) are differentiable in \( S \), except perhaps at the umbilical points \( (H^2 = K) \) of \( S \).

In the computations of this chapter, it will be convenient to write for short

\[
\langle u \wedge v, w \rangle = (u, v, w) \quad \text{for any } u, v, w \in \mathbb{R}^3.
\]

We recall that this is merely the determinant of the \( 3 \times 3 \) matrix whose columns (or lines) are the components of the vectors \( u, v, w \) in the canonical basis of \( \mathbb{R}^3 \).

**Example 1.** We shall compute the Gaussian curvature of the points of the torus covered by the parametrization (cf. Example 6 of Sec. 2.2)

\[
x(u, v) = ((a + r \cos u) \cos v, (a + r \cos u) \sin v, r \sin u),
\]

\[
0 < u < 2\pi, \quad 0 < v < 2\pi.
\]

For the computation of the coefficients \( e, f, g \), we need to know \( N \) (and thus \( x_u, x_v, x_{uv}, \) and \( x_{vu} \)):

\[
x_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u),
\]

\[
x_v = ((a + r \cos u) \sin v, (a + r \cos u) \cos v, 0),
\]

\[
x_{uv} = (-r \cos u \cos v, -r \cos u \sin v, -r \sin u),
\]

\[
x_{vu} = (r \sin u \sin v, -r \sin u \cos v, 0),
\]

\[
x_{uv} = (-r \cos u \sin v, -r \cos u \cos v, 0).
\]

From these, we obtain

\[
E = \langle x_u, x_v \rangle = r^2, \quad F = \langle x_u, x_v \rangle = 0,
\]

\[
G = \langle x_u, x_v \rangle = (a + r \cos u)^2.
\]

Introducing the values just obtained in \( e = \langle N, x_u \rangle \), we have, since \( |x_u \wedge x_v| = \sqrt{EG - F^2} \),

\[
n_{x_u} \wedge x_v = \frac{(x_u, x_v, x_{uv})}{\sqrt{EG - F^2}} = \frac{r^2(a + r \cos u)}{r(a + r \cos u)} = r.
\]

Similarly, we obtain

\[
f = \frac{(x_u, x_v, x_{uv})}{r(a + r \cos u)} = 0,
\]

\[
g = \frac{(x_u, x_v, x_{uv})}{r(a + r \cos u)} = \cos u(a + r \cos u).
\]

Finally, since \( K = \frac{(eg - f^2)(EG - F^2)}{\sqrt{EG - F^2}} \), we have that

\[
K = \frac{\cos u}{r(a + r \cos u)}.
\]

From this expression, it follows that \( K = 0 \) along the parallels \( u = \pi/2 \) and \( u = 3\pi/2 \); the points of such parallels are therefore parabolic points. In the region of the torus given by \( \pi/2 < u < 3\pi/2, K \) is negative (notice that \( r > 0 \) and \( a > r \)); the points in this region are therefore hyperbolic points. In the region given by \( 0 < u < \pi/2 \) or \( 3\pi/2 < u < 2\pi \), the curvature is positive and the points are elliptic points (Fig. 3.15).

As an application of the expression for the second fundamental form in coordinates, we shall prove a proposition which gives information about the position of a surface in the neighborhood of an elliptic or a hyperbolic point, relative to the tangent plane at this point. For instance, if we look at an elliptic point of the torus of Example 1, we find that the surface lies on one side of the tangent plane at such a point (see Fig. 3.15). On the other hand, if \( p \) is a hyperbolic point of the torus \( T \) and \( V \subset T \) is any neighborhood of \( p \), we can find points of \( V \) on both sides of \( T_p(S) \), however small \( V \) may be.
This example reflects a general local fact that is described in the following proposition.

**PROPOSITION 1.** Let \( p \in S \) be an elliptic point of a surface \( S \). Then there exists a neighborhood \( V \) of \( p \) in \( S \) such that all points in \( V \) belong to the same side of the tangent plane \( T_p(S) \). Let \( p \in S \) be a hyperbolic point. Then in each neighborhood of \( p \) there exist points of \( S \) in both sides of \( T_p(S) \).

**Proof.** Let \( x(u, v) \) be a parametrization in \( p \), with \( x(0, 0) = p \). The distance \( d \) from a point \( q = x(u, v) \) to the tangent plane \( T_p(S) \) is given by (Fig. 3-16)

\[
d = \langle x(u, v) - x(0, 0), N(p) \rangle.
\]

![Figure 3-16](image)

Since \( x(u, v) \) is differentiable, we have Taylor’s formula:

\[
x(u, v) = x(0, 0) + x_u u + x_v v + \frac{1}{2}(x_u u^2 + 2x_u u v + x_v v^2) + R,
\]

where the derivatives are taken at \((0, 0)\) and the remainder \( R \) satisfies the condition

\[
\lim_{(u, v) \to (0, 0)} \frac{R}{u^2 + v^2} = 0.
\]

It follows that

\[
d = \langle x(u, v) - x(0, 0), N(p) \rangle
\]

\[
= \frac{1}{2} [\langle x_{uu}, N(p) \rangle u^2 + 2\langle x_{uv}, N(p) \rangle u v + \langle x_{vv}, N(p) \rangle v^2] + R
\]

\[
= \frac{1}{2} (e u^2 + 2 f u v + g v^2) + R = \frac{1}{2} I_2(w) + R,
\]

where \( w = x_u + x_v \), \( R = \langle R, N(p) \rangle \), and \( \lim_{w \to 0} (R \left| w \right|^2) = 0 \).

For an elliptic point \( p \), \( I_2(w) \) has a fixed sign. Therefore, for all \((u, v)\) sufficiently near \( p \), \( d \) has the same sign as \( I_2(w) \); that is, all such \((u, v)\) belong to the same side of \( T_p(S) \).

For a hyperbolic point \( p \), in each neighborhood of \( p \) there exist points \((u, v)\) and \((\bar{u}, \bar{v})\) such that \( I_2(w) \) and \( \bar{I}_2(\bar{w}) \) have opposite signs (here \( \bar{w} = x_u + x_v \)); such points belong therefore to distinct sides of \( T_p(S) \).

Q.E.D.

No such statement as Prop. 1 can be made in a neighborhood of a parabolic or a planar point. In the above examples of parabolic and planar points (cf. Examples 3 and 6 of Sec. 3-2) the surface lies on one side of the tangent plane and may have a line in common with this plane. In the following examples we shall show that an entirely different situation may occur.

**Example 2.** The “monkey saddle” (see Fig. 3-17) is given by

\[
x = u, \quad y = v, \quad z = u^3 - 3v^2 u.
\]

A direct computation shows that at \((0, 0)\) the coefficients of the second fundamental form are \( e = f = g = 0 \); the point \((0, 0)\) is therefore a planar point. In any neighborhood of this point, however, there are points in both sides of its tangent plane.

![Figure 3-17](image)

**Example 3.** Consider the surface obtained by rotating the curve \( z = y^3 \), \(-1 < z < 1\), about the line \( z = 1 \) (see Fig. 3-18). A simple computation shows that the points generated by the rotation of the origin \( O \) are parabolic points. We shall omit this computation, because we shall prove shortly (Example 4) that the parallels and the meridians of a surface of revolution...
are lines of curvature; this, together with the fact that, for the points in question, the meridians (curves of the form \( y = x^3 \)) have zero curvature and the parallel is a normal section, will imply the above statement.

Notice that in any neighborhood of such a parabolic point there exist points in both sides of the tangent plane.

The expression of the second fundamental form in local coordinates is particularly useful for the study of the asymptotic and principal directions. We first look at the asymptotic directions.

Let \( \mathbf{x}(u, v) \) be a parametrization at \( p \in S \), with \( \mathbf{x}(0, 0) = p \), and let \( e(u, v) = e, \quad f(u, v) = f, \quad g(u, v) = g \) be the coefficients of the second fundamental form in this parametrization.

We recall that (see Def. 9 of Sec. 3-2) a connected regular curve \( C \) in the coordinate neighborhood of \( x \) is an asymptotic curve if and only if for any parametrization \( \alpha(t) = \mathbf{x}(u(t), v(t)), \quad t \in I, \) of \( C \) we have \( II(\alpha'(t)) = 0 \), for all \( t \in I \), that is, if and only if

\[
e(u')^2 + 2fuv' + g(v')^2 = 0, \quad t \in I. \tag{7}
\]

Because of that, Eq. (7) is called the differential equation of the asymptotic curves. In the next section we shall give a more precise meaning to this expression. For the time being, we want to draw from Eq. (7) only the following useful conclusion: A necessary and sufficient condition for a parametrization in a neighborhood of a hyperbolic point (\( eg - f^2 > 0 \)) to be such that the coordinate curves of the parametrization are asymptotic curves is that \( e = g = 0 \).

In fact, if both curves \( u = \text{const.}, \quad v = v(t) \) and \( u = u(t), \quad v = \text{const.} \) satisfy Eq. (7), we obtain \( e = g = 0 \). Conversely, if this last condition holds and \( f \neq 0 \), Eq. (7) becomes \( fuv' = 0 \), which is clearly satisfied by the coordinate lines.

We shall now consider the principal directions, maintaining the notations already established.

A connected regular curve \( C \) in the coordinate neighborhood of \( x \) is a line of curvature if and only if for any parametrization \( \alpha(t) = \mathbf{x}(u(t), v(t)) \) of \( C, \quad t \in I, \) we have (cf. Prop. 3 of Sec. 3-2)

\[
dN(\alpha'(t)) = \lambda(t)\alpha'(t).
\]

It follows that the functions \( u'(t), \quad v'(t) \) satisfy the system of equations

\[
\frac{fF - eG}{EG - F^2} u' + \frac{gF - fG}{EG - F^2} v' = \lambda u',
\]

\[
\frac{eF - fG}{EG - F^2} u' + \frac{fF - gE}{EG - F^2} v' = \lambda v'.
\]

By eliminating \( \lambda \) in the above system, we obtain the differential equation of the lines of curvature,

\[
(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 = 0,
\]

which may be written, in a more symmetric way, as

\[
\begin{vmatrix}
(v')^2 & -u'v' & (u')^2 \\
E & F & G \\
e & f & g
\end{vmatrix} = 0. \tag{8}
\]

Using the fact that the principal directions are orthogonal to each other, it follows easily from Eq. (8) that a necessary and sufficient condition for the coordinate curves of a parametrization to be lines of curvature in a neighborhood of a nonumbilical point is that \( F = f = 0 \).

**Example 4 (Surfaces of Revolution).** Consider a surface of revolution parametrized by (cf. Example 4 of Sec. 2-3; we have changed \( f \) and \( g \) by \( \varphi \) and \( \psi \), respectively)

\[
\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)),
\]

\[
0 < u < 2\pi, \quad a < v < b, \quad \varphi(v) \neq 0.
\]

The coefficients of the first fundamental form are given by

\[
E = \varphi^2, \quad F = 0, \quad G = (\varphi')^2 + (\psi')^2.
\]

It is convenient to assume that the rotating curve is parametrized by arc length, that is, that

\[
(\varphi')^2 + (\psi')^2 = G = 1.
\]

The computation of the coefficients of the second fundamental form is straightforward and yields

\[
e = \frac{(x_u \times x_{uv} \times x_{vu})}{\sqrt{EG - F^2}} = \frac{1}{\sqrt{EG - F^2}} \begin{vmatrix}
-\varphi \sin u & \varphi' \cos u & -\varphi \cos u \\
\varphi \cos u & \varphi' \sin u & -\varphi \sin u \\
0 & \psi' & 0
\end{vmatrix} = -\varphi \psi',
\]

\[
f = 0,
\]

\[
g = \psi' \varphi'' - \psi'' \varphi'.
\]

Since \( F = f = 0 \), we conclude that the parallels (\( v = \text{const.} \)) and the
meridians \((u = \text{const.})\) of a surface of revolution are lines of curvature of such a surface (this fact was used in Example 3).

Because

\[
K = \frac{eg - f^2}{EG - F^2} = -\frac{\psi'\psi'' - \psi\psi'}{\varphi}
\]

and \(\varphi\) is always positive, it follows that the parabolic points are given by either \(\psi' = 0\) (the tangent line to the generator curve is perpendicular to the axis of rotation) or \(\psi'\psi'' - \psi\psi'' = 0\) (the curvature of the generator curve is zero). A point which satisfies both conditions is a planar point, since these conditions imply that \(e = f = g = 0\).

It is convenient to put the Gaussian curvature in still another form. By differentiating \((\varphi')^2 + (\psi')^2 = 1\) we obtain \(\varphi'\varphi'' = -\psi'\psi''\). Thus,

\[
K = -\frac{\psi'\psi'' - \psi\psi'}{\varphi} = \frac{(\psi')^2\varphi' + (\varphi')^2\psi''}{\varphi} = -\frac{\varphi''}{\varphi}. \tag{9}
\]

Equation (9) is a convenient expression for the Gaussian curvature of a surface of revolution. It can be used, for instance, to determine the surfaces of revolution of constant Gaussian curvature (cf. Exercise 7).

To compute the principal curvatures, we first make the following general observation: If a parametrization of a regular surface is such that \(F = f = 0\), then the principal curvatures are given by \(e|E + g|G\). In fact, in this case, the Gaussian and the mean curvatures are given by (cf. Eqs. (4) and (5))

\[
K = \frac{eg}{EG}, \quad H = \frac{1}{2} \frac{eG + gE}{EG}.
\]

Since \(K\) is the product and \(2H\) is the sum of the principal curvatures, our assertion follows at once.

Thus, the principal curvatures of a surface of revolution are given by

\[
\frac{e}{E} = -\frac{\psi'}{\varphi'}, \quad \frac{g}{G} = \frac{\psi'}{\varphi'} - \frac{\psi\varphi''}{\varphi}; \tag{10}
\]

hence, the mean curvature of such a surface is

\[
H = \frac{1}{2} \frac{-\psi' + \varphi(\psi'\varphi'' - \psi\psi')}{\varphi}. \tag{11}
\]

Example 5. Very often a surface is given as the graph of a differentiable function (cf. Prop. 1, Sec. 2-2) \(z = h(x, y)\), where \((x, y)\) belong to an open set \(U \subset \mathbb{R}^2\). It is, therefore, convenient to have at hand formulas for the relevant concepts in this case. To obtain such formulas let us parametrize the surface

by

\[
x(u, v) = (u, v, h(u, v)), \quad (u, v) \in U,
\]

where \(u = x, v = y\). A simple computation shows that

\[
x_u = (1, 0, h_u), \quad x_v = (0, 1, h_v), \quad x_{uu} = (0, 0, h_{uu}), \quad x_{vv} = (0, 0, h_{vv}), \quad x_{uv} = (0, 0, h_{uv}).
\]

Thus

\[
N(x, y) = \frac{\left(-h_u, -h_v, 1\right)}{(1 + h_u^2 + h_v^2)^{1/2}}
\]

is a unit normal field on the surface, and the coefficients of the second fundamental form in this orientation are given by

\[
e = \frac{h_{xx}}{(1 + h_u^2 + h_v^2)^{1/2}},
\]

\[
f = \frac{h_{xy}}{(1 + h_u^2 + h_v^2)^{1/2}},
\]

\[
g = \frac{h_{yy}}{(1 + h_u^2 + h_v^2)^{1/2}}.
\]

From the above expressions, any needed formula can be easily computed. For instance, from Eqs. (4) and (5) we obtain the Gaussian and mean curvatures:

\[
K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_u^2 + h_v^2)^{1/2}},
\]

\[
2H = \frac{(1 + h_u^2)h_{xx} - 2h_{xy}h_{yy} + (1 + h_v^2)h_{yy}}{(1 + h_u^2 + h_v^2)^{1/2}}.
\]

There is still another, perhaps more important, reason to study surfaces given by \(z = h(x, y)\). It comes from the fact that locally any surface is the graph of a differentiable function (cf. Prop. 3, Sec. 2-2). Given a point \(p\) of a surface \(S\), we can choose the coordinate axis of \(R^3\) so that the origin \(O\) of the coordinates is at \(p\) and the axis is directed along the positive normal of \(S\) at \(p\) (thus, the \(xy\) plane agrees with \(T_p(S)\)). It follows that a neighborhood of \(p\) in \(S\) can be represented in the form \(z = h(x, y), (x, y) \in U \subset \mathbb{R}^2\), where \(U\) is an open set and \(h\) is a differentiable function (cf. Prop. 3, Sec. 2-2), with \(h(0, 0) = 0, \ h_x(0, 0) = 0, \ h_y(0, 0) = 0\) (Fig. 3-19).

The second fundamental form of \(S\) at \(p\) applied to the vector \((x, y) \in \mathbb{R}^2\) becomes, in this case,

\[
h_{xx}(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2.
\]
In elementary calculus of two variables, the above quadratic form is known as the Hessian of \( h \) at \((0, 0)\). Thus, the Hessian of \( h \) at \((0, 0)\) is the second fundamental form of \( S \) at \( p \).

Let us apply the above considerations to give a geometric interpretation of the Dupin indicatrix. With the notation as above, let \( \epsilon > 0 \) be a small number such that

\[
C = \{(x, y) \in T_p(S); h(x, y) = \epsilon\}
\]

is a regular curve (we may have to change the orientation of the surface to achieve \( \epsilon > 0 \)). We want to show that if \( p \) is not a planar point, the curve \( C \) is “approximately” similar to the Dupin indicatrix of \( S \) at \( p \) (Fig. 3-20)

To see this, let us assume further that the \( x \) and \( y \) axes are directed along the principal directions, with the \( x \) axis along the direction of maximum principal curvature. Thus, \( f = h_x(0, 0) = 0 \) and

\[
k_1(p) = \frac{\sigma}{E} = h_{xx}(0, 0), \quad k_2(p) = \frac{\zeta}{G} = h_{yy}(0, 0).
\]

By developing \( h(x, y) \) into a Taylor's expansion about \((0, 0)\), and taking into account that \( h_x(0, 0) = 0 = h_y(0, 0) \), we obtain

\[
h(x, y) = \frac{1}{2}(h_{xx}(0, 0)x^2 + 2h_{xy}(0, 0)xy + h_{yy}(0, 0)y^2) + R
\]

\[
= \frac{1}{2}(k_1x^2 + k_2y^2) + R,
\]

where

\[
\lim_{(x, y) \to (0, 0)} \frac{R}{x^2 + y^2} = 0.
\]

Thus, the curve \( C \) is given by

\[
k_1x^2 + k_2y^2 + 2R = 2\epsilon.
\]

Now, if \( p \) is not a planar point, we can consider \( k_1x^2 + k_2y^2 = 2\epsilon \) as a first-order approximation of \( C \). By using the similarity transformation,

\[
x = \tilde{x}\sqrt{2\epsilon}, \quad y = \tilde{y}\sqrt{2\epsilon},
\]

we have that \( k_1\tilde{x}^2 + k_2\tilde{y}^2 = 2\epsilon \) is transformed into the curve

\[
k_1\tilde{x}^2 + k_2\tilde{y}^2 = 1,
\]

which is the Dupin indicatrix at \( p \). This means that if \( p \) is a nonplanar point, the intersection with \( S \) of a plane parallel to \( T_p(S) \) and close to \( p \) is, in a first-order approximation, a curve similar to the Dupin indicatrix at \( p \).

If \( p \) is a planar point, this interpretation is no longer valid (cf. Exercise 11).

To conclude this section we shall give a geometrical interpretation of the Gaussian curvature in terms of the Gauss map \( N: S \to S^2 \). Actually this was how Gauss himself introduced this curvature.

To do this, we first need a definition.

Let \( S \) and \( \tilde{S} \) be two oriented regular surfaces. Let \( \varphi: S \to \tilde{S} \) be a differentiable map and assume that for some \( p \in S \), \( d\varphi \) is nonsingular. We say that \( \varphi \) is orient-
then \( \{d\varphi_p(w_1), d\varphi_p(w_2)\} \) is a positive basis in \( T_{p(S)}(S) \). If \( \{d\varphi_p(w_1), d\varphi_p(w_2)\} \) is not a positive basis, we say that \( \varphi \) is orientation-reversing at \( p \).

We now observe that both \( S \) and the unit sphere \( S^2 \) are embedded in \( R^3 \). Thus, an orientation \( N \) on \( S \) induces an orientation \( N \) in \( S^2 \). Let \( p \in S \) be such that \( dN_p \) is nonsingular. Since for a basis \( \{w_1, w_2\} \) in \( T_p(S) \)

\[
dN_p(w_1) \wedge dN_p(w_2) = \det(dN_p)(w_1 \wedge w_2) = K w_1 \wedge w_2,
\]

the Gauss map \( N \) will be orientation-preserving at \( p \in S \) if \( K(p) > 0 \) and orientation-reversing at \( p \in S \) if \( K(p) < 0 \). Intuitively, this means the following (Fig. 3-21): An orientation of \( T_p(S) \) induces an orientation of small closed curves in \( S \) around \( p \); the image by \( N \) of these curves will have the same or the opposite orientation to the initial one, depending on whether \( p \) is an elliptic or hyperbolic point, respectively.

To take this fact into account we shall make the convention that the area of a region contained in a connected neighborhood \( V \), where \( K \neq 0 \), and the area of its image by \( N \) have the same sign if \( K > 0 \) in \( V \), and opposite signs if \( K < 0 \) in \( V \) (since \( V \) is connected, \( K \) does not change sign in \( V \)).

![Figure 3-21. The Gauss map preserves orientation at an elliptic point and reverses it at a hyperbolic point.](image)

Now we can state the promised geometric interpretation of the Gaussian curvature \( K \), for \( K \neq 0 \).

**Proposition 2.** Let \( p \) be a point of a surface \( S \) such that the Gaussian curvature \( K(p) \neq 0 \), and let \( V \) be a connected neighborhood of \( p \) where \( K \) does not change sign. Then

\[
K(p) = \lim_{A \to 0} \frac{A'}{A},
\]

where \( A \) is the area of a region \( B \subset V \) containing \( p \), \( A' \) is the area of the image of \( B \) by the Gauss map \( N: S \to S^2 \), and the limit is taken through a sequence of regions \( B_n \) that converges to \( p \) in the sense that any sphere around \( p \) contains all \( B_n \), for \( n \) sufficiently large.

**Proof.** The area \( A \) of \( B \) is given by (cf. Sec. 2-5)

\[
A = \int \int_R |x_u \wedge x_v| \, du \, dv,
\]

where \( x(u, v) \) is a parametrization in \( p \), whose coordinate neighborhood contains \( V \) (\( V \) can be assumed to be sufficiently small) and \( R \) is the region in the \( uv \) plane corresponding to \( B \). The area \( A' \) of \( N(B) \) is

\[
A' = \int \int_R |N_u \wedge N_v| \, du \, dv.
\]

Using Eq. (1), the definition of \( K \), and the above convention, we can write

\[
A' = \int \int_R K |x_u \wedge x_v| \, du \, dv. \tag{12}
\]

Going to the limit and denoting also by \( R \) the area of the region \( R \), we obtain

\[
\lim_{A \to 0} \frac{A'}{A} = \lim_{R \to 0} \frac{A'/R}{A/R} = \lim_{R \to 0} \frac{(1/R) \int \int_R K |x_u \wedge x_v| \, du \, dv}{(1/R) \int \int_R |x_u \wedge x_v| \, du \, dv} = \frac{K}{|x_u \wedge x_v|} = K
\]

(notice that we have used the mean value theorem for double integrals), and this proves the proposition. \( \text{Q.E.D.} \)

**Remark.** Comparing the proposition with the expression of the curvature

\[
k = \lim_{s \to 0} \frac{\sigma}{s},
\]

we see that

\[
K(p) = k(p).
\]
of a plane curve $C$ at $p$ (here $s$ is the arc length of a small segment of $C$ containing $p$, and $\sigma$ is the arc length of its image in the indicatrix of tangents; cf. Exercise 3 of Sec. 1-5), we see that the Gaussian curvature $K$ is the analogue, for surfaces, of the curvature $k$ of plane curves.

**EXERCISES**

1. Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

2. Determine the asymptotic curves and the lines of curvature of the helicoid $x = u \cos v, y = u \sin v, z = cu$, and show that their mean curvature is zero.

3. Determine the asymptotic curves of the catenoid
   $$x(u, v) = (\cosh v \cos u, \cosh v \sin u, v).$$

4. Determine the asymptotic curves and the lines of curvature of $z = xy$.

5. Consider the parametrized surface (Enneper's surface)
   $$x(u, v) = \left( u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + uv^2, u^2 - v^2 \right)$$
   and show that
   a. The coefficients of the first fundamental form are
      $$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$  
   b. The coefficients of the second fundamental form are
      $$e = 2, \quad g = -2, \quad f = 0.$$  
   c. The principal curvatures are
      $$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$  
   d. The lines of curvature are the coordinate curves.
   e. The asymptotic curves are $u + v = \text{const.}, u - v = \text{const}.$

6. (A Surface with $K \equiv -1$; the Pseudosphere.)
   a. Determine an equation for the plane curve $C$, which is such that the segment of the tangent line between the point of tangency and some line $r$ in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).
   b. Rotate the tractrix $C$ about the line $r$; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

7. (Surfaces of Revolution with Constant Curvature.) $(\varphi(v) \cos u, \varphi(v) \sin u, \psi(v))$ is given as a surface of revolution with constant Gaussian curvature $K$. To determine the functions $\varphi$ and $\psi$, choose the parameter $v$ in such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that $v$ is the arc length of the generating curve $(\varphi(v), \psi(v))$). Show that
   a. $\varphi$ satisfies $\varphi'' + K \varphi = 0$ and $\psi$ is given by $\psi = \int \sqrt{1 - (\varphi')^2} \, dv$; thus, $0 < u < 2\pi$, and the domain of $v$ is such that the last integral makes sense.
   b. All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane $xOy$ are given by
      $$\varphi(v) = C \cos v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v} \, dv,$$
      where $C$ is a constant ($C = \varphi(0)$). Determine the domain of $v$ and draw a rough sketch of the profile of the surface in the $xz$ plane for the cases $C = 1, C > 1, C < 1$. Observe that $C = 1$ gives a sphere (Fig. 3-23).
   c. All surfaces of revolution with constant curvature $K = -1$ may be given by one of the following types:
      1. $\varphi(v) = C \cosh v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sinh^2 v} \, dv.$

---

Figure 3-22. The pseudosphere.

Figure 3-23