1 Curves

1-1. Introduction

The differential geometry of curves and surfaces has two aspects. One, which may be called classical differential geometry, started with the beginnings of calculus. Roughly speaking, classical differential geometry is the study of local properties of curves and surfaces. By local properties we mean those properties which depend only on the behavior of the curve or surface in the neighborhood of a point. The methods which have shown themselves to be adequate in the study of such properties are the methods of differential calculus. Because of this, the curves and surfaces considered in differential geometry will be defined by functions which can be differentiated a certain number of times.

The other aspect is the so-called global differential geometry. Here one studies the influence of the local properties on the behavior of the entire curve or surface. We shall come back to this aspect of differential geometry later in the book.

Perhaps the most interesting and representative part of classical differential geometry is the study of surfaces. However, some local properties of curves appear naturally while studying surfaces. We shall therefore use this first chapter for a brief treatment of curves.

The chapter has been organized in such a way that a reader interested mostly in surfaces can read only Secs. 1-2 through 1-5. Sections 1-2 through 1-4 contain essentially introductory material (parametrized curves, arc length, vector product), which will probably be known from other courses and is included here for completeness. Section 1-5 is the heart of the chapter
and contains the material of curves needed for the study of surfaces. For those wishing to go a bit further on the subject of curves, we have included Secs. 1-6 and 1-7.

1-2. Parametrized Curves

We denote by $R^3$ the set of triples $(x, y, z)$ of real numbers. Our goal is to characterize certain subsets of $R^3$ (to be called curves) that are, in a certain sense, one-dimensional and to which the methods of differential calculus can be applied. A natural way of defining such subsets is through differentiable functions. We say that a real function of a real variable is differentiable (or smooth) if it has, at all points, derivatives of all orders (which are automatically continuous). A first definition of curve, not entirely satisfactory but sufficient for the purposes of this chapter, is the following.

**Definition.** A parametrized differentiable curve is a differentiable map \( \alpha : I \to R^3 \) of an open interval \( I = (a, b) \) of the real line \( R \) into \( R^3 \).

The word differentiable in this definition means that \( \alpha \) is a correspondence which maps each \( t \in I \) into a point \( \alpha(t) = (x(t), y(t), z(t)) \in R^3 \) in such a way that the functions \( x(t), y(t), z(t) \) are differentiable. The variable \( t \) is called the parameter of the curve. The word interval is taken in a generalized sense, so that we do not exclude the cases \( a = -\infty, b = +\infty \).

If we denote by \( x'(t) \) the first derivative of \( x \) at the point \( t \) and use similar notations for the functions \( y \) and \( z \), the vector \( (x'(t), y'(t), z'(t)) = \alpha'(t) \in R^3 \) is called the tangent vector (or velocity vector) of the curve \( \alpha \) at \( t \). The image set \( \alpha(I) \subset R^3 \) is called the trace of \( \alpha \). As illustrated by Example 5 below, one should carefully distinguish a parametrized curve, which is a map, from its trace, which is a subset of \( R^3 \).

A warning about terminology. Many people use the term “infinitely differentiable” for functions which have derivatives of all orders and reserve the word “differentiable” to mean that only the existence of the first derivative is required. We shall not follow this usage.

**Example 1.** The parametrized differentiable curve given by

\[
\alpha(t) = (a \cos t, a \sin t, bt), \quad t \in R,
\]

has as its trace in \( R^3 \) a helix of pitch \( 2\pi b \) on the cylinder \( x^2 + y^2 = a^2 \). The parameter \( t \) here measures the angle which the \( x \) axis makes with the line joining the origin \( 0 \) to the projection of the point \( \alpha(t) \) over the \( xy \) plane (see Fig. 1-1).

**Example 2.** The map \( \alpha : R \to R^2 \) given by \( \alpha(t) = (t^3, t^2) \), \( t \in R \), is a parametrized differentiable curve which has Fig. 1-2 as its trace. Notice that \( \alpha'(0) = (0, 0) \); that is, the velocity vector is zero for \( t = 0 \).

**Example 3.** The map \( \alpha : R \to R^2 \) given by \( \alpha(t) = (t^3 - 4t, t^2 - 4) \), \( t \in R \), is a parametrized differentiable curve (see Fig. 1-3). Notice that \( \alpha(2) = \alpha(-2) = (0, 0) \); that is, the map \( \alpha \) is not one-to-one.

**Example 4.** The map \( \alpha : R \to R^2 \) given by \( \alpha(t) = (t, |t|) \), \( t \in R \), is not a parametrized differentiable curve, since \( |t| \) is not differentiable at \( t = 0 \) (Fig. 1-4).

**Example 5.** The two distinct parametrized curves

\[
\begin{align*}
\alpha(t) &= (\cos t, \sin t), \\
\beta(t) &= (\cos 2t, \sin 2t),
\end{align*}
\]
where \( t \in (0 - \varepsilon, 2\pi + \varepsilon), \varepsilon > 0 \), have the same trace, namely, the circle \( x^2 + y^2 = 1 \). Notice that the velocity vector of the second curve is the double of the first one (Fig. 1-5).

![Figure 1-5](image)

We shall now recall briefly some properties of the inner (or dot) product of vectors in \( \mathbb{R}^3 \). Let \( u = (u_1, u_2, u_3) \in \mathbb{R}^3 \) and define its norm (or length) by

\[ |u| = \sqrt{u_1^2 + u_2^2 + u_3^2}. \]

Geometrically, \( |u| \) is the distance from the point \((u_1, u_2, u_3)\) to the origin \((0, 0, 0)\). Now, let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) belong to \( \mathbb{R}^3 \), and let \( \theta \), \( 0 \leq \theta \leq \pi \), be the angle formed by the segments \( 0u \) and \( 0v \). The inner product \( u \cdot v \) is defined by (Fig. 1-6)

\[ u \cdot v = |u||v| \cos \theta. \]

![Figure 1-6](image)

The following properties hold:

1. Assume that \( u \) and \( v \) are nonzero vectors. Then \( u \cdot v = 0 \) if and only if \( u \) is orthogonal to \( v \).

2. \( u \cdot v = v \cdot u \).

3. \( \lambda(u \cdot v) = \lambda u \cdot v = u \cdot \lambda v \).

4. \( u \cdot (v + w) = u \cdot v + u \cdot w \).

A useful expression for the inner product can be obtained as follows. Let \( e_1 = (1, 0, 0) \), \( e_2 = (0, 1, 0) \), and \( e_3 = (0, 0, 1) \). It is easily checked that \( e_i \cdot e_j = 1 \) if \( i = j \) and that \( e_i \cdot e_j = 0 \) if \( i \neq j \), where \( i, j = 1, 2, 3 \). Thus, by writing

\[ u = u_1 e_1 + u_2 e_2 + u_3 e_3, \quad v = v_1 e_1 + v_2 e_2 + v_3 e_3, \]

and using properties 3 and 4, we obtain

\[ u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3. \]

From the above expression it follows that if \( u(t) \) and \( v(t) \), \( t \in I \), are differentiable curves, then \( u(t) \cdot v(t) \) is a differentiable function, and

\[ \frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t). \]

EXERCISES

1. Find a parametrized curve \( \alpha(t) \) whose trace is the circle \( x^2 + y^2 = 1 \) such that \( \alpha(t) \) runs clockwise around the circle with \( \alpha(0) = (0, 1) \).

2. Let \( \alpha(t) \) be a parametrized curve which does not pass through the origin. If \( \alpha(t_0) \) is the point of the trace of \( \alpha \) closest to the origin and \( \alpha'(t_0) \neq 0 \), show that the position vector \( \alpha(t_0) \) is orthogonal to \( \alpha'(t_0) \).

3. A parametrized curve \( \alpha(t) \) has the property that its second derivative \( \alpha''(t) \) is identically zero. What can be said about \( \alpha \)?

4. Let \( \alpha : I \to \mathbb{R}^3 \) be a parametrized curve and let \( v \in \mathbb{R}^3 \) be a fixed vector. Assume that \( \alpha'(t) \) is orthogonal to \( v \) for all \( t \in I \) and that \( \alpha(t) \) is also orthogonal to \( v \). Prove that \( \alpha(t) \) is orthogonal to \( v \) for all \( t \in I \).

5. Let \( \alpha : I \to \mathbb{R}^3 \) be a parametrized curve, with \( \alpha'(t) \neq 0 \) for all \( t \in I \). Show that \( |\alpha(t)| \) is a nonzero constant if and only if \( \alpha(t) \) is orthogonal to \( \alpha'(t) \) for all \( t \in I \).

1-3. Regular Curves; Arc Length

Let \( \alpha : I \to \mathbb{R}^3 \) be a parametrized differentiable curve. For each \( t \in I \) where \( \alpha'(t) \neq 0 \), there is a well-defined straight line, which contains the point \( \alpha(t) \) and the vector \( \alpha'(t) \). This line is called the tangent line to \( \alpha \) at \( t \). For the studen
of the differential geometry of a curve it is essential that there exists such a tangent line at every point. Therefore, we call any point \( t \) where \( \alpha'(t) = 0 \) a singular point of \( \alpha \) and restrict our attention to curves without singular points. Notice that the point \( t = 0 \) in Example 2 of Sec. 1-2 is a singular point.

**DEFINITION.** A parametrized differentiable curve \( \alpha : I \rightarrow \mathbb{R}^3 \) is said to be regular if \( \alpha'(t) \neq 0 \) for all \( t \in I \).

From now on we shall consider only regular parametrized differentiable curves (and, for convenience, shall usually omit the word differentiable).

Given \( t \in I \), the arc length of a regular parametrized curve \( \alpha : I \rightarrow \mathbb{R}^3 \), from the point \( t_0 \), is by definition

\[
s(t) = \int_{t_0}^{t} \left| \alpha'(t) \right| \, dt,
\]

where

\[
\left| \alpha'(t) \right| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}
\]

is the length of the vector \( \alpha'(t) \). Since \( \alpha'(t) \neq 0 \), the arc-length \( s \) is a differentiable function of \( t \) and \( ds/dt = |\alpha'(t)| \).

In Exercise 8 we shall present a geometric justification for the above definition of arc length.

It can happen that the parameter \( t \) is already the arc length measured from some point. In this case, \( ds/dt = 1 = |\alpha'(t)| \); that is, the velocity vector has constant length equal to 1. Conversely, if \( |\alpha'(t)| \equiv 1 \), then

\[
s = \int_{t_0}^{t} dt = t - t_0;
\]

i.e., \( t \) is the arc length of \( \alpha \) measured from some point.

To simplify our exposition, we shall restrict ourselves to curves parametrized by arc length; we shall see later (see Sec. 1-5) that this restriction is not essential. In general, it is not necessary to mention the origin of the arc length \( s \), since most concepts are defined only in terms of the derivatives of \( \alpha(s) \).

It is convenient to set still another convention. Given the curve \( \alpha \) parametrized by arc length \( s \in (a, b) \), we may consider the curve \( \beta \) defined in \( (-b, -a) \) by \( \beta(-s) = \alpha(s) \), which has the same trace as the first one but is described in the opposite direction. We say, then, that these two curves differ by a change of orientation.

### EXERCISES

1. Show that the tangent lines to the regular parametrized curve \( \alpha(t) = (3t, 3t^2, 2t^3) \) make a constant angle with the line \( y = 0, z = x \).

2. A circular disk of radius 1 in the plane \( xy \) rolls without slipping along the \( x \) axis. The figure described by a point of the circumference of the disk is called a cycloid (Fig. 1-7).

![Figure 1-7. The cycloid.](image)

*a.* Obtain a parametrized curve \( \alpha : R \rightarrow R^3 \) the trace of which is the cycloid, and determine its singular points.

*b.* Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

3. Let \( 0A = 2a \) be the diameter of a circle \( S^1 \) and \( 0Y \) and \( AV \) be the tangents to \( S^1 \) at \( 0 \) and \( A \), respectively. A half-line \( r \) is drawn from \( 0 \) which meets the circle \( S^1 \) at \( C \) and the line \( AV \) at \( B \). On \( 0B \) mark off the segment \( 0p = CB \). If we rotate \( r \) about \( 0 \), the point \( p \) will describe a curve called the cissoid of Diocles. By taking \( 0A \) as the \( x \) axis and \( 0Y \) as the \( y \) axis, prove that

\[
\alpha(t) = \left( \frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2} \right), \quad t \in R,
\]

is the cissoid of Diocles \( t = \tan \theta \); see Fig. 1-8).

*b.* The origin \( (0, 0) \) is a singular point of the cissoid.

*c.* As \( t \rightarrow \infty \), \( \alpha(t) \) approaches the line \( x = 2a \), and \( \alpha'(t) \rightarrow (0, 2a) \). Thus, as \( t \rightarrow \infty \), the curve and its tangent approach the line \( x = 2a \); we say that \( x = 2a \) is an asymptote to the cissoid.

4. Let \( \alpha : (0, \pi) \rightarrow R^2 \) be given by

\[
\alpha(t) = \left( \sin t, \cos t + \log \tan \frac{t}{2} \right),
\]

where \( t \) is the angle that the \( y \) axis makes with the vector \( \alpha'(t) \). The trace of \( \alpha \) is called the tractrix (Fig. 1-9). Show that
a. $\alpha$ is a differentiable parametrized curve, regular except at $t = \pi/2$.

b. The length of the segment of the tangent of the tractrix between the point of tangency and the $y$ axis is constantly equal to 1.

5. Let $\alpha : (-1, +\infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left( \frac{3at}{1+t^3}, \frac{3at^2}{1+t^3} \right).$$

Prove that:

a. For $t = 0$, $\alpha$ is tangent to the $x$ axis.

b. As $t \to +\infty$, $\alpha(t) \to (0, 0)$ and $\alpha'(t) \to (0, 0)$.

c. Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approach the line $x + y + a = 0$.

The figure obtained by completing the trace of $\alpha$ in such a way that it becomes symmetric relative to the line $y = x$ is called the folium of Descartes (see Fig. 1-10).

6. Let $\alpha(t) = (ae^{it}, ae^{-it}), t \in \mathbb{R}$, $a$ and $b$ constants, $a > 0, b < 0$, be a parametrized curve.

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\lim_{t \to +\infty} \int_{t_0}^{t} \|\alpha'(t)\| \, dt
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is finite; that is, $\alpha$ has finite arc length in $[t_0, \infty)$.

Figure 1-10. Folium of Descartes.

Figure 1-11. Logarithmic spiral.
7. A map \( \alpha : I \to \mathbb{R}^3 \) is called a curve of class \( C^k \) if each of \( k \) coordinates functions in the expression \( \alpha(t) = (x(t), y(t), z(t)) \) has continuous derivatives up to order \( k \). If \( \alpha \) is merely continuous, we say that \( \alpha \) is of class \( C^0 \). A curve \( \alpha \) is called simple if the map \( \alpha \) is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let \( \alpha : I \to \mathbb{R}^3 \) be a simple curve of class \( C^0 \). We say that \( \alpha \) has a weak tangent at \( t = t_0 \in I \) if the line determined by \( \alpha(t_0 + h) \) and \( \alpha(t_0) \) has a limit position when \( h \to 0 \). We say that \( \alpha \) has a strong tangent at \( t = t_0 \) if the line determined by \( \alpha(t_0 + h) \) and \( \alpha(t_0 + k) \) has a limit position when \( h, k \to 0 \). Show that

a. \( \alpha(t) = (t^3, t^2), t \in \mathbb{R} \), has a weak tangent but not a strong tangent at \( t = 0 \).

b. If \( \alpha : I \to \mathbb{R}^3 \) is of class \( C^1 \) and regular at \( t = t_0 \), then it has a strong tangent at \( t = t_0 \).

c. The curve given by

\[
\alpha(t) = \begin{cases} (t^3, t^2), & t \geq 0, \\ (t^2, -t^3), & t \leq 0, \end{cases}
\]

is of class \( C^1 \) but not of class \( C^2 \). Draw a sketch of the curve and its tangent vectors.

8. Let \( \alpha : I \to \mathbb{R}^2 \) be a differentiable curve and let \([a, b] \subset I \) be a closed interval. For every partition

\[
a = t_0 < t_1 < \cdots < t_n = b
\]

of \([a, b]\), consider the sum \( \sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| \) = \( l(\alpha, P) \), where \( P \) stands for the given partition. The norm \( |P| \) of a partition \( P \) is defined as

\[
|P| = \max(t_i - t_{i-1}), \quad i = 1, \ldots, n.
\]

Geometrically, \( l(\alpha, P) \) is the length of a polygon inscribed in \( \alpha([a, b]) \) with vertices in \( \alpha(t_i) \) (see Fig. 1-12). The point of the exercise is to show that the arc length of \( \alpha([a, b]) \) is, in some sense, a limit of lengths of inscribed polygons.

![Figure 1-12](image)

The Vector Product in \( \mathbb{R}^3 \)

In this section, we shall present some properties of the vector product in \( \mathbb{R}^3 \). They will be found useful in our later study of curves and surfaces.

It is convenient to begin by reviewing the notion of orientation of a vector space. Two ordered bases \( e = \{e_i\} \) and \( f = \{f_i\}, i = 1, \ldots, n \), of an \( n \)-dimensional vector space \( V \) have the same orientation if the matrix of change of basis has positive determinant. We denote this relation by \( e \sim f \). From elementary properties of determinants, it follows that \( e \sim f \) is an equivalence relation; i.e., it satisfies

1. \( e \sim e \).
2. If \( e \sim f \), then \( f \sim e \).
3. If \( e \sim f, f \sim g \), then \( e \sim g \).

Prove that given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( |P| < \delta \) then

\[
\left| \int_{a}^{b} |\alpha'(t)| \, dt - l(\alpha, P) \right| < \epsilon.
\]
The set of all ordered bases of $V$ is thus decomposed into equivalence classes (the elements of a given class are related by $\sim$) which by property 3 are disjoint. Since the determinant of a change of basis is either positive or negative, there are only two such classes.

Each of the equivalence classes determined by the above relation is called an orientation of $V$. Therefore, $V$ has two orientations, and if we fix one of them arbitrarily, the other one is called the opposite orientation.

In the case $V = \mathbb{R}^3$, there exists a natural ordered basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and we shall call the orientation corresponding to this basis the positive orientation of $\mathbb{R}^3$, the other one being the negative orientation (of course, this applies equally well to any $\mathbb{R}^n$). We also say that a given ordered basis of $\mathbb{R}^3$ is positive (or negative) if it belongs to the positive (or negative) orientation of $\mathbb{R}^3$. Thus, the ordered basis $e_1, e_2, e_3$ is a negative basis, since the matrix which changes this basis into $e_1, e_2, e_3$ has determinant equal to $-1$.

We now come to the vector product. Let $u, v \in \mathbb{R}^3$. The vector product of $u$ and $v$ (in that order) is the unique vector $u \wedge v \in \mathbb{R}^3$ characterized by

$$(u \wedge v) \cdot w = \det(u, v, w) \quad \text{for all } w \in \mathbb{R}^3.$$ 

Here $\det(u, v, w)$ means that if we express $u$, $v$, and $w$ in the natural basis $\{e_i\}$,

$$u = \sum u_i e_i, \quad v = \sum v_i e_i,$$

$$w = \sum w_i e_i, \quad i = 1, 2, 3,$$

then

$$\det(u, v, w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

where $|a_{ij}|$ denotes the determinant of the matrix $(a_{ij})$. It is immediate from the definition that

$$u \wedge v = \begin{vmatrix} u_2 & u_3 & e_1 \\ u_1 & u_3 & e_2 \\ u_1 & u_2 & e_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ u_1 & u_2 \end{vmatrix} e_3 + \begin{vmatrix} u_1 & u_3 \\ u_1 & u_2 \end{vmatrix} e_2 - \begin{vmatrix} u_1 & u_2 \\ u_1 & u_3 \end{vmatrix} e_1.$$  \hspace{1cm} (I)

**Remark.** It is also very frequent to write $u \wedge v$ as $u \times v$ and refer to it as the cross product.

The following properties can easily be checked (actually they just express the usual properties of determinants):

1. $u \wedge v = -v \wedge u$ (anticommutativity).
2. $u \wedge v$ depends linearly on $u$ and $v$; i.e., for any real numbers $a, b$, we have

$$\text{(au + bw) \wedge v = au \wedge v + bw \wedge v.}$$

3. $u \wedge v = 0$ if and only if $u$ and $v$ are linearly dependent.
4. $(u \wedge v) \cdot u = 0$, $(u \wedge v) \cdot v = 0$.

It follows from property 4 that the vector product $u \wedge v \neq 0$ is normal to a plane generated by $u$ and $v$. To give a geometric interpretation of its norm and its direction, we proceed as follows.

First, we observe that $(u \wedge v) \cdot (u \wedge v) = |u \wedge v|^2 > 0$. This means that the determinant of the vectors $u, v, u \wedge v$ is positive; that is, $(u, v, u \wedge v)$ is a positive basis.

Next, we prove the relation

$$(u \wedge v) \cdot (x \wedge y) = \begin{vmatrix} u \cdot x & u \cdot y & v \cdot x \\ u \cdot y & v \cdot y & v \cdot x \end{vmatrix},$$

where $u, v, x, y$ are arbitrary vectors. This can easily be done by observing that both sides are linear in $u, v, x, y$. Thus, it suffices to check that

$$(e_i \wedge e_j) \cdot (e_k \wedge e_l) = \begin{vmatrix} e_i \cdot e_k & e_i \cdot e_k \\ e_i \cdot e_l & e_i \cdot e_l \end{vmatrix} = 0$$

for all $i, j, k, l = 1, 2, 3$. This is a straightforward verification.

It follows that

$$|u \wedge v|^2 = |u \cdot u|^2 + |u \cdot v|^2 - 2|u|^2 |v|^2 (1 - \cos^2 \theta) = A^2,$$

where $\theta$ is the angle of $u$ and $v$, and $A$ is the area of a parallelogram generated by $u$ and $v$.

In short, the vector product of $u$ and $v$ is a vector $u \wedge v$ perpendicular to a plane generated by $u$ and $v$, with a norm equal to the area of a parallelogram generated by $u$ and $v$ and a direction such that $(u, v, u \wedge v)$ is a positive basis (Fig. 1-13).

![Figure 1-13](image-url)
The vector product is not associative. In fact, we have the following identity:

$$ (u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u, $$

which can be proved as follows. First we observe that both sides are linear in $u, v, w$; thus, the identity will be true if it holds for all basis vectors. This last verification is, however, straightforward; for instance,

$$ (e_1 \wedge e_2) \wedge e_1 = e_2 = (e_1 \cdot e_2)e_2 = (e_2 \cdot e_1)e_1. $$

Finally, let $u(t) = (u_1(t), u_2(t), u_3(t))$ and $v(t) = (v_1(t), v_2(t), v_3(t))$ be differentiable maps from the interval $(a, b)$ to $\mathbb{R}^3, r \in (a, b)$. It follows immediately from Eq. (1) that $u(t) \wedge v(t)$ is also differentiable and that

$$ \frac{d}{dt} (u(t) \wedge v(t)) = \frac{du}{dt} \wedge v(t) + u(t) \wedge \frac{dv}{dt}. $$

Vector products appear naturally in many geometrical constructions. Actually, most of the geometry of planes and lines in $\mathbb{R}^3$ can be neatly expressed in terms of vector products and determinants. We shall review some of this material in the following exercises.

**EXERCISES**

1. Check whether the following bases are positive:
   a. The basis [(1, 3), (4, 2)] in $\mathbb{R}^2.$
   b. The basis [(1, 3, 5), (2, 3, 7), (4, 8, 3)] in $\mathbb{R}^3$.

2. A plane $P$ contained in $\mathbb{R}^3$ is given by the equation $ax + by + cz + d = 0.$ Show that the vector $v = (a, b, c)$ is perpendicular to the plane and that $|d|/\sqrt{a^2 + b^2 + c^2}$ measures the distance from the plane to the origin $(0, 0, 0)$.

3. Given two planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0,$ prove that a necessary and sufficient condition for them to be parallel is

$$ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}, $$

where the convention is made that if a denominator is zero, the corresponding numerator is also zero (we say that two planes are parallel if they either coincide or do not intersect).

4. Show that the equation of a plane passing through three noncollinear points $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2), p_3 = (x_3, y_3, z_3)$ is given by

$$ (p - p_1) \wedge (p - p_2)(p - p_3) = 0, $$

where $p = (x, y, z)$ is an arbitrary point of the plane and $p - p_1,$ for instance, means the vector $(x - x_1, y - y_1, z - z_1)$.

5. Given two nonparallel planes $a_i x + b_i y + c_i z + d_i = 0, i = 1, 2,$ show that their line of intersection may be parameterized as

$$ x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t, $$

where $(x_0, y_0, z_0)$ belongs to the intersection and $u = (u_1, u_2, u_3)$ is the vector product $u = v_1 \wedge v_2,$ $v_i = (a_i, b_i, c_i), i = 1, 2.$

6. Prove that a necessary and sufficient condition for the plane

$$ ax + by + cz + d = 0 $$

and the line $x - x_0 = u_1 t, y - y_0 = u_2 t, z - z_0 = u_3 t$ to be parallel is

$$ au_1 + bu_2 + cu_3 = 0. $$

7. Prove that the distance $d$ between the nonparallel lines

$$ x - x_0 = u_1 t, \quad y - y_0 = u_2 t, \quad z - z_0 = u_3 t, $$

is given by

$$ d = \frac{|(u_1 \wedge v_1) \cdot r|}{|u_1 \wedge v_1|}, $$

where $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3), r = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$.

8. Determine the angle of intersection of the plane $3x + 4y + 7z + 8 = 0$ and the line $x - 2 = 3t, y - 3 = 5t, z - 5 = 9t$.

9. The natural orientation of $\mathbb{R}^2$ makes it possible to associate a sign to the area $A$ of a parallelogram generated by two linearly independent vectors $u, v \in \mathbb{R}^2.$ To do this, let $[e_1], i = 1, 2,$ be the natural ordered basis of $\mathbb{R}^2$, and write $u = u_1 e_1 + u_2 e_2, v = v_1 e_1 + v_2 e_2.$ Observe the matrix relation

$$ (u \wedge v) \cdot r = (u_1 \wedge v_1) (u_2 \wedge v_2) $$

and conclude that

$$ A^2 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2. $$

Since the last determinant has the same sign as the basis $[u, v]$, we can say that $A$ is positive or negative according to whether the orientation of $[u, v]$ is positive or negative. This is the oriented area in $\mathbb{R}^2$. 

- The Vector Product in $\mathbb{R}^3$
11. a. Show that the volume $V$ of a parallelepiped generated by three linearly independent vectors $u, v, w \in \mathbb{R}^3$ is given by $V = \lvert (u \wedge v) \cdot w \rvert$, and introduce an oriented volume in $\mathbb{R}^3$.

b. Prove that

$$ V^2 = \begin{vmatrix} u \cdot u & u \cdot v & u \cdot w \\ v \cdot u & v \cdot v & v \cdot w \\ w \cdot u & w \cdot v & w \cdot w \end{vmatrix} $$

12. Given the vectors $v \neq 0$ and $w$, show that there exists a vector $u$ such that $u \wedge v = w$ if and only if $v$ is perpendicular to $w$. Is this vector $u$ uniquely determined? If not, what is the most general solution?

13. Let $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ be differentiable maps from the interval $(a, b)$ into $\mathbb{R}^3$. If the derivatives $\alpha'(t)$ and $\beta'(t)$ satisfy the conditions

$$ \alpha'(t) = a\alpha(t) + b\beta(t), \quad \beta'(t) = c\alpha(t) - \alpha(t), $$

where $a, b,$ and $c$ are constants, show that $u(t) \wedge v(t)$ is a constant vector.

14. Find all unit vectors which are perpendicular to the vector $(2, 2, 1)$ and parallel to the plane determined by the points $(0, 0, 0), (1, -2, 1), (-1, 1, 1)$.

**1-5. The Local Theory of Curves Parametrized by Arc Length**

This section contains the main results of curves which will be used in the later parts of the book.

Let $\alpha: I = (a, b) \rightarrow \mathbb{R}^3$ be a curve parametrized by arc length $s$. Since the tangent vector $\alpha'(s)$ has unit length, the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at $s$. $|\alpha''(s)|$ gives, therefore, a measure of how rapidly the curve pulls away from the tangent line at $s$, in a neighborhood of $s$ (see Fig. 1-14). This suggests the following definition.

**DEFINITION.** Let $\alpha: I \rightarrow \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The number $|\alpha''(s)| = k(s)$ is called the curvature of $\alpha$ at $s$.

If $\alpha$ is a straight line, $\alpha(s) = us + v$, where $u$ and $v$ are constant vectors ($|u| = 1$), then $k = 0$. Conversely, if $k = |\alpha''(s)| = 0$, then by integration $\alpha(s) = us + v$, and the curve is a straight line.

Notice that by a change of orientation, the tangent vector changes its direction; that is, if $\beta(-s) = \alpha(s)$, then

$$ \frac{d\beta}{ds}(-s) = -\frac{d\alpha}{ds}(s). $$

Therefore, $\alpha''(s)$ and the curvature remain invariant under a change of orientation.

At points where $k(s) \neq 0$, a unit vector $n(s)$ in the direction $\alpha''(s)$ is well defined by the equation $\alpha''(s) = k(s)n(s)$. Moreover, $\alpha''(s)$ is normal to $\alpha'(s)$, because by differentiating $\alpha'(s) \cdot \alpha'(s) = 1$ we obtain $\alpha'(s) \cdot \alpha''(s) = 0$. Thus, $n(s)$ is normal to $\alpha'(s)$ and is called the normal vector at $s$. The plane determined by the unit tangent and normal vectors, $\alpha'(s)$ and $n(s)$, is called the osculating plane at $s$. (See Fig. 1-15.)

At points where $k(s) = 0$, the normal vector (and therefore the osculating plane) is not defined (cf. Exercise 10). To proceed with the local analysis of curves, we need, in an essential way, the osculating plane. It is therefore
convenient to say that \( s \in I \) is a singular point of order 1 if \( a''(s) = 0 \) (in this context, the points where \( a'(s) = 0 \) are called singular points of order 0).

In what follows, we shall restrict ourselves to curves parametrized by arc length without singular points of order 1. We shall denote by \( t(s) = a'(s) \) the unit tangent vector of \( a \) at \( s \). Thus, \( t'(s) = k(s)n(s) \).

The unit vector \( b(s) = t(s) \wedge n(s) \) is normal to the osculating plane and will be called the binormal vector at \( s \). Since \( b(s) \) is a unit vector, the length \( |b(s)| \) measures the rate of change of the neighboring osculating planes with the osculating plane at \( s \); that is, \( b(s) \) measures how rapidly the curve pulls away from the osculating plane at \( s \), in a neighborhood of \( s \) (see Fig. 1-15).

To compute \( b(s) \) we observe that, on the one hand, \( b(s) \) is normal to \( b(s) \) and that, on the other hand,

\[
b'(s) = t'(s) \wedge n(s) + t(s) \wedge n'(s) = t(s) \wedge n'(s);
\]

that is, \( b'(s) \) is normal to \( t(s) \). It follows that \( b'(s) \) is parallel to \( n(s) \), and we may write

\[
b'(s) = \tau(s)n(s)
\]

for some function \( \tau(s) \). (Warning: Many authors write \(-\tau(s)\) instead of our \( \tau(s) \).)

**DEFINITION.** Let \( a: I \rightarrow \mathbb{R}^3 \) be a curve parametrized by arc length \( s \) such that \( a''(s) \neq 0 \), \( s \in I \). The number \( \tau(s) \) defined by \( b'(s) = \tau(s)n(s) \) is called the torsion of \( a \) at \( s \).

If \( a \) is a plane curve (that is, \( a(I) \) is contained in a plane), then the plane of the curve agrees with the osculating plane; hence, \( \tau \equiv 0 \). Conversely, if \( \tau \equiv 0 \) (and \( k \neq 0 \)), we have that \( b(s) = b_0 = \text{constant} \), and therefore

\[
(a(s) \cdot b_0)' = a'(s) \cdot b_0 = 0.
\]

It follows that \( a(s) \cdot b_0 = \text{constant} \); hence, \( a(s) \) is contained in a plane normal to \( b_0 \). The condition that \( k \neq 0 \) everywhere is essential here. In Exercise 10 we shall give an example where \( \tau \) can be defined to be identically zero and yet the curve is not a plane curve.

In contrast to the curvature, the torsion may be either positive or negative. The sign of the torsion has a geometric interpretation, to be given later (Sec. 1-6).

Notice that by changing orientation the binormal vector changes sign, since \( b = t \wedge n \). It follows that \( b'(s) \), and, therefore, the torsion, remains invariant under a change of orientation.

Let us summarize our position. To each value of the parameter \( s \), we have associated three orthogonal unit vectors \( t(s), n(s), b(s) \). The trihedron thus formed is referred to as the Frenet trihedron at \( s \). The derivatives \( t'(s) = kn, b'(s) = \tau n \) of the vectors \( t(s) \) and \( b(s) \), when expressed in the basis \( \{t, n, b\} \), yield geometrical entities (curvature \( k \) and torsion \( \tau \) which give us information about the behavior of \( a \) in a neighborhood of \( s \).

The search for other local geometrical entities would lead us to compute \( n'(s) \). However, since \( n = b \wedge t \), we have

\[
n'(s) = b'(s) \wedge t(s) + b(s) \wedge t'(s) = -\tau b - kt,
\]

and we obtain again the curvature and the torsion.

For later use, we shall call the equations

\[
t' = kn, \quad n' = -kt + \tau b, \quad b' = \tau n
\]

the Frenet formulas (we have omitted the \( s \), for convenience). In this context, the following terminology is usual. The \( tb \) plane is called the rectifying plane, and the \( nb \) plane the normal plane. The lines which contain \( n(s) \) and \( b(s) \) and pass through \( a(s) \) are called the principal normal and the binormal, respectively. The inverse \( R = 1/k \) of the curvature is called the radius of curvature at \( s \). Of course, a circle of radius \( r \) has radius of curvature equal to \( r \), as one can easily verify.

Physically, we can think of a curve in \( \mathbb{R}^3 \) as being obtained from a straight line by bending (curvature) and twisting (torsion). After reflecting on this construction, we are led to conjecture the following statement, which, roughly speaking, shows that \( k \) and \( \tau \) describe completely the local behavior of the curve.

**FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES.** Given differentiable functions \( k(s) > 0 \) and \( \tau(s) \), \( s \in I \), there exists a regular parametrized curve \( a: I \rightarrow \mathbb{R}^3 \) such that \( s \) is the arc length, \( k(s) \) is the curvature, and \( \tau(s) \) is the torsion of \( a \). Moreover, any other curve \( \tilde{a} \), satisfying the same conditions, differs from \( a \) by a rigid motion; that is, there exists an orthogonal linear map \( p \) of \( \mathbb{R}^3 \), with positive determinant, and a vector \( c \) such that \( \tilde{a} = p \circ a + c \).

The above statement is true. A complete proof involves the theorem of existence and uniqueness of solutions of ordinary differential equations and will be given in the appendix to Chap. 4. A proof of the uniqueness, up to
rigid motions, of curves having the same $s$, $k(s)$, and $\tau(s)$ is, however, simple and can be given here.

**Proof of the Uniqueness Part of the Fundamental Theorem.** We first remark that arc length, curvature, and torsion are invariant under rigid motions; that means, for instance, that if $M: R^3 \rightarrow R^3$ is a rigid motion and $\alpha = \alpha(t)$ is a parametrized curve, then

$$\int_{s_0}^{s_1} \frac{d\alpha}{dt} \ dt = \int_{s_0}^{s_1} \frac{d(M \circ \alpha)}{dt} \ dt.$$  

That is plausible, since these concepts are defined by using inner or vector products of certain derivatives (the derivatives are invariant under translations, and the inner and vector products are expressed by means of lengths and angles of vectors, and thus also invariant under rigid motions). A careful checking can be left as an exercise (see Exercise 6).

Now, assume that two curves $\alpha = \alpha(s)$ and $\tilde{\alpha} = \tilde{\alpha}(s)$ satisfy the conditions $k(s) = \tilde{k}(s)$ and $\tau(s) = \tilde{\tau}(s)$, $s \in I$. Let $t_0, \ n_0, \ b_0$ and $\tilde{t}_0, \ \tilde{n}_0, \ \tilde{b}_0$ be the Frenet trihedrons at $s = s_0 \in I$ of $\alpha$ and $\tilde{\alpha}$, respectively. Clearly, there is a rigid motion which takes $\tilde{\alpha}(s_0)$ into $\alpha(s_0)$ and $\tilde{t}_0, \ \tilde{n}_0, \ \tilde{b}_0$ into $t_0, \ n_0, \ b_0$. Thus, after performing this rigid motion on $\tilde{\alpha}$, we have that $\tilde{\alpha}(s_0) = \alpha(s_0)$ and that the Frenet trihedrons $t(s), \ n(s), \ b(s)$ and $\tilde{t}(s), \ \tilde{n}(s), \ \tilde{b}(s)$ of $\alpha$ and $\tilde{\alpha}$, respectively, satisfy the Frenet equations:

$$\frac{dt}{ds} = k n$$  
$$\frac{dn}{ds} = -k t + \tau b$$  
$$\frac{db}{ds} = \tau n$$

with $t(s_0) = \tilde{t}(s_0)$, $n(s_0) = \tilde{n}(s_0)$, $b(s_0) = \tilde{b}(s_0)$.

We now observe, by using the Frenet equations, that

$$\frac{1}{2} \frac{d}{ds} \left[ |t - \tilde{t}|^2 + |n - \tilde{n}|^2 + |b - \tilde{b}|^2 \right]$$  
$$= \langle t - \tilde{t}, \ t - \tilde{t} \rangle + \langle b - \tilde{b}, b - \tilde{b} \rangle + \langle n - \tilde{n}, n - \tilde{n} \rangle$$  
$$= k \langle t - \tilde{t}, n - \tilde{n} \rangle + \tau \langle b - \tilde{b}, n - \tilde{n} \rangle - k \langle n - \tilde{n}, t - \tilde{t} \rangle$$  
$$- \tau \langle n - \tilde{n}, b - \tilde{b} \rangle$$  
$$= 0$$

for all $s \in I$. Thus, the above expression is constant, and, since it is zero for $s = s_0$, it is identically zero. It follows that $t(s) = \tilde{t}(s)$, $n(s) = \tilde{n}(s)$, $b(s) = \tilde{b}(s)$ for all $s \in I$. Since

$$\frac{dt}{ds} = t = \tilde{t} = \frac{d\tilde{t}}{ds},$$

we obtain $(d/ds) (\alpha - \tilde{\alpha}) = 0$. Thus, $\alpha(s) = \tilde{\alpha}(s) + a$, where $a$ is a constant vector. Since $\alpha(s_0) = \tilde{\alpha}(s_0)$, we have $a = 0$; hence, $\alpha(s) = \tilde{\alpha}(s)$ for all $s \in I$. Q.E.D.

**Remark 1.** In the particular case of a plane curve $\alpha: I \rightarrow \mathbb{R}^2$, it is possible to give the curvature $k$ a sign. For that, let $[e_1, e_2]$ be the natural basis (see Sec. 1-4) of $\mathbb{R}^2$ and define the normal vector $n(s), s \in I$, by requiring the basis $\{t(s), n(s)\}$ to have the same orientation as the basis $\{e_1, e_2\}$. The curvature $k$ is then defined by

$$\frac{dt}{ds} = k n$$

and might be either positive or negative. It is clear that $|k|$ agrees with the previous definition and that $k$ changes sign when we change either the orientation of $\alpha$ or the orientation of $\mathbb{R}^3$ (Fig. 1-16).

![Figure 1-16](image)

It should also be remarked that, in the case of plane curves ($\tau = 0$), the proof of the fundamental theorem, referred to above, is actually very simple (see Exercise 9).

**Remark 2.** Given a regular parametrized curve $\alpha: I \rightarrow \mathbb{R}^3$ (not necessarily parametrized by arc length), it is possible to obtain a curve $\beta: I \rightarrow \mathbb{R}^3$ parametrized by arc length which has the same trace as $\alpha$. In fact, let
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\[ s = s(t) = \int_{t_0}^{t} |\alpha'(t)| \, dt, \quad t, t_0 \in I. \]

Since \( ds/dt = |\alpha'(t)| \neq 0 \), the function \( s = s(t) \) has a differentiable inverse \( t = t(s), s \in s(I) = J \), where, by abuse of notation, \( t \) also denotes the inverse function \( s^{-1} \) of \( s \). Now set \( \beta = \alpha \circ t : J \to R^3 \). Clearly, \( \beta(J) = \alpha(I) \) and \( |\beta'(s)| = |\alpha'(t) \cdot dt(ds)| = 1 \). This shows that \( \beta \) has the same trace as \( \alpha \) and is parametrized by arc length. It is usual to say that \( \beta \) is a reparametrization of \( \alpha(I) \) by arc length.

This fact allows us to extend all local concepts previously defined to regular curves with an arbitrary parameter. Thus, we say that the curvature \( k(t) \) of \( \alpha : I \to R^3 \) at \( t \in I \) is the curvature of a reparametrization \( \beta : J \to R^3 \) of \( \alpha(I) \) by arc length at the corresponding point \( s = s(t) \). This is clearly independent of the choice of \( \beta \) and shows that the restriction, made at the end of Sec. 1-3, of considering only curves parametrized by arc length is not essential.

In applications, it is often convenient to have explicit formulas for the geometrical entities in terms of an arbitrary parameter; we shall present some of them in Exercise 12.

**EXERCISES**

Unless explicitly stated, \( \alpha : I \to R^3 \) is a curve parametrized by arc length \( s \), with curvature \( k(s) \neq 0 \), for all \( s \in I \).

1. Given the parametrized curve (helix)
   \[
   \alpha(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, b \frac{s}{c} \right), \quad s \in R,
   \]
   where \( c^2 = a^2 + b^2 \),
   a. Show that the parameter \( s \) is the arc length.
   b. Determine the curvature and the torsion of \( \alpha \).
   c. Determine the osculating plane of \( \alpha \).
   d. Show that the lines containing \( n(s) \) and passing through \( \alpha(s) \) meet the \( z \) axis under a constant angle equal to \( \pi/2 \).
   e. Show that the tangent lines to \( \alpha \) make a constant angle with the \( z \) axis.

2. Show that the torsion \( \tau \) of \( \alpha \) is given by
   \[
   \tau(s) = \frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.
   \]
   3. Assume that \( \alpha(I) \subset R^2 \) (i.e., \( \alpha \) is a plane curve) and give \( k \) a sign as in the text. Transport the vectors \( t(s) \) parallel to themselves in such a way that the origins of \( t(s) \) agree with the origin of \( R^2 \); the end points of \( t(s) \) then describe a parametrized curve \( s \to t(s) \) called the indicatrix of tangents of \( \alpha \). Let \( \theta(s) \) be the angle from \( e_1 \) to \( t(s) \) in the orientation of \( R^2 \). Prove (a) and (b) (notice that we are assuming that \( k \neq 0 \)).
   a. The indicatrix of tangents is a regular parametrized curve.
   b. \( dt/ds = (d\theta/ds)n \), that is, \( k = d\theta/ds \).

4. Assume that all normals of a parametrized curve pass through a fixed point. Prove that the trace of the curve is contained in a circle.

5. A regular parametrized curve \( \alpha \) has the property that all its tangent lines pass through a fixed point.
   a. Prove that the trace of \( \alpha \) is a (segment of a) straight line.
   b. Does the conclusion in part a still hold if \( \alpha \) is not regular?

6. A translation by a vector \( v \) in \( R^3 \) is the map \( A : R^3 \to R^3 \) that is given by \( A(p) = p + v \), \( p \in R^3 \). A linear map \( p : R^3 \to R^3 \) is an orthogonal transformation when \( pu \cdot pv = u \cdot v \) for all vectors \( u, v \in R^3 \). A rigid motion in \( R^3 \) is the result of composing a translation with an orthogonal transformation with positive determinant (this last condition is included because we expect rigid motions to preserve orientation).
   a. Demonstrate that the norm of a vector and the angle \( \theta \) between two vectors, \( 0 \leq \theta \leq \pi \), are invariant under orthogonal transformations with positive determinant.
   b. Show that the vector product of two vectors is invariant under orthogonal transformations with positive determinant. Is the assertion still true if we drop the condition on the determinant?
   c. Show that the arc length, the curvature, and the torsion of a parametrized curve are (whenever defined) invariant under rigid motions.

7. Let \( \alpha : I \to R^2 \) be a regular parametrized plane curve (arbitrary parameter), and define \( n = n(t) \) and \( k = k(t) \) as in Remark 1. Assume that \( k(t) \neq 0 \), \( t \in I \). In this situation, the curve
   \[
   \beta(t) = \alpha(t) + \frac{1}{k(t)} n(t), \quad t \in I,
   \]
   is called the evolute of \( \alpha \) (Fig. 1-17).
   a. Show that the tangent at \( t \) of the evolute of \( \alpha \) is the normal to \( \alpha \) at \( t \).
   b. Consider the normal lines of \( \alpha \) at two neighboring points \( t_1, t_2, t_1 \neq t_2 \). Let \( t_1 \) approach \( t_2 \) and show that the intersection points of the normals converge to a point on the trace of the evolute of \( \alpha \).

8. The trace of the parametrized curve (arbitrary parameter)
   \[
   \alpha(t) = (t, \cosh t), \quad t \in R,
   \]
   is called the catenary.
Curves Parametrized by Arc Length

a. Show that the signed curvature (cf. Remark 1) of the catenary is
\[ k(t) = \frac{1}{\cosh^2 t}. \]

b. Show that the evolute (cf. Exercise 7) of the catenary is
\[ \beta(t) = (t - \sinh t \cosh t, 2 \cosh t, 2 \sinh t). \]

9. Given a differentiable function \( k(s), s \in I, \) show that the parametrized plane curve having \( k(s) = k \) as curvature is given by
\[ \alpha(s) = \left( \int \cos \theta(s) \, ds + a, \int \sin \theta(s) \, ds + b \right), \]
where
\[ \theta(s) = \int k(s) \, ds + \varphi, \]
and that the curve is determined up to a translation of the vector \((a, b)\) and a rotation of the angle \( \varphi. \)

10. Consider the map
\[ \alpha(t) = \begin{cases} (t, 0, e^{-1/t^a}), & t > 0 \smallskip \\
           (t, e^{-1/t^a}, 0), & t < 0 \smallskip \\
           (0, 0, 0), & t = 0 \end{cases} \]
a. Prove that \( \alpha \) is a differentiable curve.
b. Prove that \( \alpha \) is regular for all \( t \) and that the curvature \( k(t) \neq 0, \) for \( t \neq 0, \)
\( t \neq \pm \sqrt[3]{3}, \) and \( k(0) = 0. \)

c. Show that the limit of the osculating planes as \( t \to 0, t > 0, \) is the plane \( y = 0 \) but that the limit of the osculating planes as \( t \to 0, t < 0, \) is the plane \( z = 0 \) (this implies that the normal vector is discontinuous at \( t = 0 \) and shows why we excluded points where \( k = 0 \)).
d. Show that \( \tau \) can be defined so that \( \tau \equiv 0, \) even though \( \alpha \) is not a plane curve.

11. One often gives a plane curve in polar coordinates by \( \rho = \rho(\theta), a \leq \theta \leq b. \)
a. Show that the arc length is
\[ \int_a^b \sqrt{\rho^2 + (\rho')^2} \, d\theta, \]
where the prime denotes the derivative relative to \( \theta. \)
b. Show that the curvature is
\[ k(\theta) = \frac{2(\rho')^2 - \rho \rho'' + \rho^2}{(\rho^2 + \rho'^2)^{3/2}}. \]

12. Let \( \alpha: I \to \mathbb{R}^3 \) be a regular parametrized curve (not necessarily by arc length) and let \( \beta: J \to \mathbb{R}^3 \) be a reparametrization of \( \alpha(I) \) by the arc length \( s = \xi(t), \) measured from \( t_0 \in I \) (see Remark 2). Let \( t = t(s) \) be the inverse function of \( s \) and set \( d\xi/ds = \alpha', \ d^2\xi/dt^2 = \alpha', \) etc. Prove that
a. \( dt/ds = 1/|\alpha'|, \ d^2t/ds^2 = -(\alpha' \cdot \alpha'')/|\alpha'|^3. \)
b. The curvature of \( \alpha \) at \( t \in I \) is
\[ k(t) = \frac{\alpha' \wedge \alpha''}{|\alpha'|^3}. \]
c. The torsion of \( \alpha \) at \( t \in I \) is
\[ \tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}. \]
d. If \( \alpha: I \to \mathbb{R}^2 \) is a plane curve \( \alpha(t) = (x(t), y(t)) \), the signed curvature (see Remark 1) of \( \alpha \) at \( t \) is
\[ k(t) = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}. \]

13. Assume that \( \tau(s) \neq 0 \) and \( k'(s) \neq 0 \) for all \( s \in I. \) Show that a necessary and sufficient condition for \( \alpha(I) \) to lie on a sphere is that
\[ R^2 + (R')^2 T^2 = \text{const.}, \]
where \( R = 1/k, \ T = 1/\tau, \) and \( R' \) is the derivative of \( R \) relative to \( s. \)

14. Let \( \alpha: (a, b) \to \mathbb{R}^2 \) be a regular parametrized plane curve. Assume that there exists \( t_0, a < t_0 < b, \) such that the distance \( |\alpha(t)| \) from the origin to the trace of
\( \alpha \) will be a maximum at \( t_0 \). Prove that the curvature \( k \) of \( \alpha \) at \( t_0 \) satisfies 
\[ |k(t_0)| \geq 1/|\alpha(t_0)|. \]

**15.** Show that the knowledge of the vector function \( b = b(s) \) (binormal vector) of a curve \( \alpha \), with nonzero torsion everywhere, determines the curvature \( k(s) \) and the absolute value of the torsion \( \tau(s) \) of \( \alpha \).

**16.** Show that the knowledge of the vector function \( n = n(s) \) (normal vector) of a curve \( \alpha \), with nonzero torsion everywhere, determines the curvature \( k(s) \) and the torsion \( \tau(s) \) of \( \alpha \).

17. In general, a curve \( \alpha \) is called a helix if the tangent lines of \( \alpha \) make a constant angle with a fixed direction. Assume that \( \tau(s) \neq 0, s \in I \), and prove that:

* a. \( \alpha \) is a helix if and only if \( k/\tau = \text{const} \).

* b. \( \alpha \) is a helix if and only if the lines containing \( n(s) \) and passing through \( \alpha(s) \) are parallel to a fixed plane.

* c. \( \alpha \) is a helix if and only if the lines containing \( b(s) \) and passing through \( \alpha(s) \) make a constant angle with a fixed direction.

The curve
\[
\alpha(s) = \left( \frac{a}{c} \int \sin \theta(s) \, ds, \frac{a}{c} \int \cos \theta(s) \, ds, \frac{b}{c} s \right),
\]

where \( c^2 = a^2 + b^2 \), is a helix, and that \( k/\tau = a/b \).

**18.** Let \( \alpha: I \to \mathbb{R}^3 \) be a parametrized regular curve (not necessarily by arc length) with \( k(t) \neq 0, \tau(t) \neq 0, t \in I \). The curve \( \alpha \) is called a Bertrand curve if there exists a curve \( \tilde{\alpha}: I \to \mathbb{R}^3 \) such that the normal lines of \( \alpha \) and \( \tilde{\alpha} \) at \( t \in I \) are equal. In this case, \( \tilde{\alpha} \) is called a Bertrand mate of \( \alpha \), and we can write
\[
\tilde{\alpha}(t) = \alpha(t) + rt(t).
\]

Prove that

* a. \( r \) is constant.

* b. \( \alpha \) is a Bertrand curve if and only if there exists a linear relation
\[
Ak(t) + Br(t) = 1, \quad t \in I,
\]

where \( A, B \) are nonzero constants and \( k \) and \( \tau \) are the curvature and torsion of \( \alpha \), respectively.

* c. If \( \alpha \) has more than one Bertrand mate, it has infinitely many Bertrand mates. This case occurs if and only if \( \alpha \) is a circular helix.

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1-6. **The Local Canonical Form**

One of the most effective methods of solving problems in geometry consists of finding a coordinate system which is adapted to the problem. In the study of local properties of a curve, in the neighborhood of the point \( s \), we have a natural coordinate system, namely the Frenet trihedron at \( s \). It is therefore convenient to refer the curve to this trihedron.

Let \( \alpha: I \to \mathbb{R}^3 \) be a curve parametrized by arc length without singular points of order 1. We shall write the equations of the curve, in a neighborhood of \( s_0 \), using the trihedron \( t(s_0), n(s_0), b(s_0) \) as a basis for \( \mathbb{R}^3 \). We may assume, without loss of generality, that \( s_0 = 0 \), and we shall consider the (finite) Taylor expansion
\[
\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + R,
\]
where \( \lim_{s \to 0}\frac{R}{s^3} = 0 \). Since \( \alpha'(0) = t, \alpha''(0) = kn \), and
\[
\alpha'''(0) = (kn)' = k'n + kn' = k'n - k^2t - k\tau b,
\]
we obtain
\[
\alpha(s) - \alpha(0) = \left( s - \frac{k^2s^3}{3!} \right)t + \left( \frac{s^2k}{2} + \frac{s^3k}{3!} \right)n - \frac{s^3}{3!}k\tau b + R,
\]
where all terms are computed at \( s = 0 \).

Let us now take the system \( Oxyz \) in such a way that the origin \( O \) agrees with \( \alpha(0) \) and that \( t = (1, 0, 0), n = (0, 1, 0), b = (0, 0, 1) \). Under these conditions, \( \alpha(s) = (x(s), y(s), z(s)) \) is given by

\[
x(s) = s - \frac{k^2s^3}{6} + R_x,
\]
\[
y(s) = \frac{k}{2}s^2 + \frac{k's^3}{6} + R_y,
\]
\[
z(s) = -\frac{k\tau}{6}s^3 + R_z,
\]

where \( R = (R_x, R_y, R_z) \). The representation (1) is called the local canonical form of \( \alpha \), in a neighborhood of \( s = 0 \). In Fig. 1-18 is a rough sketch of the projections of the trace of \( \alpha \), for \( s \) small, in the \( in, tb, \) and \( nb \) planes.

†This section may be omitted on a first reading.
The helix of Exercise 1 of Sec. 1-5 has negative torsion. An example of a curve with positive torsion is the helix

$$\mathbf{a}(s) = \left( a \cos \frac{s}{c}, a \sin \frac{s}{c}, -b \frac{s}{c} \right)$$

obtained from the first one by a reflection in the xz plane (see Fig. 1-19).

Remark. It is also usual to define torsion by $b^1 = -\mathbf{t} \mathbf{n}$. With such a definition, the torsion of the helix of Exercise 1 becomes positive.

Another consequence of the canonical form is the existence of a neighborhood $J \subset I$ of $s = 0$ such that $a(J)$ is entirely contained in the one side of the rectifying plane toward which the vector $n$ is pointing (see Fig. 1-18). In fact, since $k > 0$, we obtain, for $s$ sufficiently small, $y(s) \geq 0$, and $y(s) = 0$ if and only if $s = 0$. This proves our claim.

As a last application of the canonical form, we mention the following property of the oscillating plane. The oscillating plane at $s$ is the limit position of the plane determined by the tangent line at $s$ and the point $a(s + h)$ when $h \to 0$. To prove this, let us assume that $s = 0$. Thus, every plane containing the tangent at $s = 0$ is of the form $z = cy$ or $y = 0$. The plane $y = 0$ is the rectifying plane that, as seen above, contains no points near $a(0)$ (except $a(0)$ itself) and that may therefore be discarded from our considerations. The condition for the plane $z = cy$ to pass through $a(s + h)$ is $(s = 0)$

$$c = \frac{z(h)}{y(h)} = \frac{-k}{6} t h^3 + \cdots$$

Letting $h \to 0$, we see that $c \to 0$. Therefore, the limit position of the plane $z = c(h)y(s)$ is the plane $z = 0$, that is, the osculating plane, as we wished.

**EXERCISES**

1. Let $a: I \to \mathbb{R}^3$ be a curve parametrized by arc length with curvature $k(s) \neq 0$, $s \in I$. Let $P$ be a plane satisfying both of the following conditions:

   1. $P$ contains the tangent line at $s$.
   2. Given any neighborhood $J \subset I$ of $s$, there exist points of $a(J)$ in both sides of $P$.

   Prove that $P$ is the osculating plane of $a$ at $s$.

2. Let $a: I \to \mathbb{R}^3$ be a curve parametrized by arc length, with curvature $k(s) \neq 0$, $s \in I$. Show that