CSE291
Topics in Computer Graphics
Mesh Animation
Matthias Zwicker
University of California, San Diego
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Paper assignments
1. Iman Mostafavi, Pose space deformation
2. Will Chang, Laplacian Surface Editing, Poisson Mesh Editing
3. Wan-Yen Lo, Linear Rotation-Invariant Coordinates for Meshes
4. Chih Liang, Mesh-based IK
5. Karen Lin, Building Efficient, Accurate Character Skins from Examples
6. Alex Zavodny, A Morphable Model For The Synthesis Of 3D Faces
7. Mike Caloud, Meshless deformations based on shape matching
8. Eric Hill, Elastically deformable models
9. Patrick Shyu, Stable real time deformations
10. Henrick Shyu, Discrete shells
11. Steve Rotenberg, Graphical modeling and animation of ductile fracture

Curves
Frenet frame
The three orthogonal unit vectors \( t(s), n(s), b(s) \)
Frenet formulas
\[
\begin{align*}
\dot{t} &= \kappa n \\
\dot{n} &= -\kappa t - \tau b \\
\dot{b} &= \tau n
\end{align*}
\]

Curves
Fundamental theorem
For functions \( k(s), \tau(s) \) there exists a curve \( \alpha(s) \) such that \( s \) is the arc length, \( k(s) \) is the curvature, and \( \tau(s) \) is the torsion of \( \alpha \).
Any other curve satisfying these conditions differs from \( \alpha \) by a rigid motion.

Differential
• Given
  - Map \( F : U \subset \mathbb{R}^n \to \mathbb{R}^m \)
    \[
    F(x_1, \ldots, x_n) = \begin{bmatrix}
    f_1(x_1, \ldots, x_n) \\
    \vdots \\
    f_m(x_1, \ldots, x_n)
    \end{bmatrix}
    \]
  - Curve \( \alpha : (-\epsilon, \epsilon) \to U \)
    \( \alpha(0) = p, \alpha'(0) = \dot{w} \)
  - Curve \( \beta = F \circ \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m \)
Differential

- Given
  - Map \( F : U \subset \mathbb{R}^n \to \mathbb{R}^m \)
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  \]
  - Curve \( \alpha : (-\varepsilon, \varepsilon) \to U, \quad \alpha(0) = p, \alpha'(0) = w \)
  - Curve \( \beta = F \circ \alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^m \)

Differential of the map \( F \)
is defined as \( dF_p(w) = \beta'(0) \)

Differential

- Is a linear function
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix} w
\]

independent of parameterization of \( \alpha \)

- This matrix is called the Jacobian

- Differential is a generalization of directional derivative

Tangent space

- Given a local parameterization \( \mathbf{x}(u, v) \) of \( S \)

Tangent space

The subspace of dimension 2 \( d\mathbf{x}\mathbb{R}^2 \subset \mathbb{R}^3 \)

First fundamental form

- The quadratic form \( L_p(w) = (\mathbf{w}, \mathbf{w})_p = |\mathbf{w}|^2 \geq 0, \ \mathbf{w} \in T_p(S) \)
is called the first fundamental form of \( S \) at \( p \)

- Express it in the basis \( \{ \mathbf{x}_u, \mathbf{x}_v \} \)
\[
L_p(\mathbf{x}_u \mathbf{v}^\top + \mathbf{x}_v \mathbf{u}^\top) = E|\mathbf{v}|^2 + 2F \mathbf{u} \mathbf{v}^\top + G|\mathbf{u}|^2
\]

- \( E = (\mathbf{x}_u, \mathbf{x}_u) \)
- \( F = (\mathbf{x}_u, \mathbf{x}_v) \)
- \( G = (\mathbf{x}_v, \mathbf{x}_v) \)

First fundamental form

- Intrinsic property of a surface

- Useful to compute arc lengths, angles, area
### Gauss map

![Gauss map diagram](image)

### Differential of Gauss map

\[ dN_p : T_p(S) \rightarrow T_p(S) \]
\[ dN_p(v) = N'(0) \]

### Today

- Second fundamental form
- Curvatures
- Laplacian operator and minimal surface

### Second fundamental form

- Defined as the quadratic form \( \mathbf{II}_p \) in \( T_p(S) \) by

\[ \mathbf{II}_p = -\langle dN_p, v \rangle \]

### Interpretation

- Note

\[ \langle N(s), \alpha'(s) \rangle = 0 \Rightarrow \langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle \]

- Therefore

\[ \mathbf{II}_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \]
Interpretation

• Note
\( \langle N(s), \alpha'(s) \rangle = 0 \Rightarrow \langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle \)
• Therefore
\[
\Pi_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N'(0), \alpha'(0) \rangle = \langle N(0), \alpha''(0) \rangle
\]

\[
\Pi_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle
\]
\[
= -\langle N'(0), \alpha'(0) \rangle - \langle N(0), \alpha''(0) \rangle
\]
\[
= \langle N, kn \rangle(p) = k \cos \theta = k_n(p)
\]

where \( \alpha''(0) = kn, \quad \langle N, n \rangle = \cos \theta \)
• \( k_n(p) \) is called the normal curvature of the curve \( \alpha \)

Interpretation

Proposition
All curves in \( S \) with the same tangent line at a point \( p \in S \) have the same normal curvatures

Normal section of \( S \) at \( p \) along \( \nu \)
Curve defined by intersection of \( S \) with the plane defined by the normal at \( p \) and a tangent direction \( \nu \)

Note
Second fundamental form is the curvature of the normal section

Questions?
**Principal curvatures**

• The maximum normal curvature $k_1$ and the minimum normal curvature $k_2$ are called the principal curvatures.
• The corresponding tangent directions $e_1, e_2$ are called principal directions.

**Principal curvatures**

**Properties**

• The principal directions are the eigenvectors of the differential of the Gauss map.
  $$dN_p(e_i) = -k_1 e_1, dN_p(e_2) = -k_2 e_2$$
• The principal directions form an orthonormal basis for $T_p(S)$.

**Principal directions**

- Direction of maximum curvature
- Direction of minimum curvature

[Diewald, Rumpf]

**Principal directions**

- Direction of maximum curvature

[Diewald, Rumpf]

**Principal directions**

- Which one is minimum/maximum?

[Diewald, Rumpf]

**Euler formula**

- The second fundamental form expressed in the basis $e_1, e_2$.
- For a unit vector $v = e_1 \cos \theta + e_2 \sin \theta$
  $$II_p(v) = k_1 \cos^2 \theta + k_2 \sin^2 \theta$$
Gaussian and mean curvature

Gaussian curvature \( K \)
The determinant of \( dN_p \)

Mean curvature \( H \)
Half the negative of the trace of \( dN_p \)

In terms of principal curvatures
\[
K = k_1 k_2, \quad H = \frac{k_1 + k_2}{2}
\]

Taxonomy

A point on the surface is

- Elliptic if \( \text{det}(dN_p) > 0 \), i.e., \( k_1 k_2 > 0 \)
- Hyperbolic if \( \text{det}(dN_p) < 0 \), i.e., \( k_1 k_2 < 0 \)
- Parabolic if
  \[
  \text{det}(dN_p) = 0, dN_p \neq 0, \text{ i.e., } k_1 = 0 \text{ or } k_2 = 0
  \]
- Planar if \( dN_p = 0 \), i.e., \( k_1 = k_2 = 0 \)
- Umbilical if \( k_1 = k_2 \)

In local coordinates

- Express second fundamental form using a local parameterization
  \[
  x: U \subset \mathbb{R}^2 \rightarrow S \\
x(u, v) = p \in S
  \]
- Basis for tangent space \( x_u, x_v \)
  Tangent vectors \( x_u u' + x_v v' \)
- What is \( II_p(u', v'), dN_p(u', v') \)

Questions?

- Given curve \( \alpha(t) \) with \( \alpha' = x_u u' + x_v v' \) at \( p \)
- Per definition
  \[
  dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'
  \]
In local coordinates

- Given curve $a(t)$ with $a' = x_u u' + x_v v'$ at $p$
- Per definition
  \[ dN(a') = N'(a(t), v(t)) = N_u u' + N_v v' \]
- Plug into second fundamental form
  \[
  \text{II}_p(a') = -\langle dN(a'), a' \rangle \\
  = -\langle N_u u' + N_v v', x_u u' + x_v v' \rangle \\
  = a(u')^2 + 2f u' v' + g(v')^2
  \]

Weingarten equations

- Because $N_u, N_v \in T_p(S)$ we can write
  \[ N_u = a_{11} x_u + a_{21} x_v \]
  \[ N_v = a_{12} x_u + a_{22} x_v \]
  and $dN$ in the basis $x_u, x_v$
  \[ dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} \]
- The coefficients $a_{11}, a_{21}, a_{12}, a_{22}$ express the differential of the Gauss map in the basis $x_u, x_v$ of $T_p(S)$

Weingarten equations

- With coefficients $E, F, G$ of first fundamental form

Weingarten equations

- Note $\langle N, x_u \rangle = \langle N, x_v \rangle = 0$
- Therefore
  \[ \begin{align*}
  a &= -\langle N_u, x_u \rangle = \langle N, x_u \rangle, \\
  f &= -\langle N_u, x_v \rangle = \langle N, x_v \rangle = -\langle N_v, x_v \rangle, \\
  g &= -\langle N_v, x_u \rangle = \langle N, x_u \rangle,
  \end{align*} \]

Curvatures

Remember
- Curvatures are defined in terms of the differential of the Gauss map
- Gaussian curvature: determinant of $dN$
  \[ K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2} \]
- Mean curvature: negative half of trace of $dN$
- Principal curvatures: eigenvalues of $dN$
Gaussian curvature revisited

Geometric definition

\[ K(p) = \lim_{A \to 0} \frac{A'}{A} \]

- A area of neighborhood of \( p \)
- \( A' \) area of Gauss map of neighborhood of \( p \)
- Equivalent to previous definition, prove see Do Carmo, page 167

Minimal surfaces

Definition

A surface is minimal if its mean curvature vanishes everywhere

Proposition

Given boundaries, a minimal surface is the surface with minimum surface area respecting the boundaries (proof Do Carmo, page 201)

Mean curvature normal

Definition (mean curvature normal) \( H = BN \)

Definition

A parameteritzation is isothermal if

\[ \langle x_u, x_u \rangle = \langle x_v, x_v \rangle = \lambda^2 \quad \text{and} \quad \langle x_u, x_v \rangle = 0 \]

Proposition

If \( x \) is isothermal, then

\[ x_{uu} + x_{vv} = 2\lambda^2 H \]

Laplace operator

Definition

Laplace operator (Laplacian)

\[ \Delta x(u, v) = x_{uu} + x_{vv} \]

- A surface with an isothermal parameterization is minimal, if the Laplacian of the coordinate functions vanishes everywhere

- Then, the coordinate functions are called harmonic
<table>
<thead>
<tr>
<th>Questions?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Class projects</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Proposal by October 5</td>
</tr>
<tr>
<td>• Paper you want to work on</td>
</tr>
<tr>
<td>• Plan for implementation</td>
</tr>
<tr>
<td>• Libraries, etc.</td>
</tr>
<tr>
<td>• Data</td>
</tr>
<tr>
<td>• Progress report by October 26</td>
</tr>
<tr>
<td>• Wiki</td>
</tr>
</tbody>
</table>