Monte Carlo History

- Use Random numbers (therefore the name)
- Early use in neutron transport
- von Neumann
- Metropolis

Monte Carlo Algorithms

The good:
- Flexible
- Easy to implement
- Often easy to understand
- Robust in complex domains
- Efficient for high-dimensional problems

Monte Carlo Algorithms

The bad:
- Noisy (variance)
- Slow convergence (requires many samples)

Random Variables

Random variable: \( X \)
Probability distribution function (PDF): \( p(x) \)

\[
\int_{-\infty}^{\infty} p(x) \, dx = 1
\]
**Random Variables**

Cumulative probability distribution function: $P(x)$

$$P(x) = \int_{-\infty}^{x} p(\mu) d\mu$$

$P(a \leq X \leq b) = \int_{a}^{b} p(x) dx$

**Expected Value**

Expected value: $E\{X\}$

$$E\{X\} = \int x p(x) dx$$

$$E\{X + Y\} = E\{X\} + E\{Y\}$$

**Expected Value**

$$E\{X\} \approx \frac{1}{N} \sum_{i=1}^{N} x_i$$

where $x_i$ are distributed according to $p(x)$

Probability $\left[ E\{X\} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} x_i \right] = 1$

**Example: Random Numbers**

If $\xi$ is a uniformly distributed random number between 0 and 1 then the PDF $p(x)$ is:

$$p(x) = \begin{cases} 1 & \text{if } 0 \leq \xi \leq 1 \\ 0 & \text{else} \end{cases}$$

$$E\{X\} = \int_{-\infty}^{0} x p(x) dx = 0.5$$

$$P(x < 0.7) = \int_{-\infty}^{0.7} p(x) dx = \int_{0}^{0.7} 1 dx = 0.7$$

**Monte Carlo Integration**

We want to compute the value $I$ of an integral

$$I = \int_{0}^{1} f(x) dx$$

$$E\{f(X)\} = \int f(x) p(x) dx \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

$$I = \int f(x) dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$
Monte Carlo Integration

double integrate( int N )
{
    double x, sum=0.0;
    for (int i=0; i<N; i++) {
        x = drand48();
        sum += exp(sin(3*x*x));
    }
    return sum / double(N);
}

Monte Carlo Integration

\[ f(x) = e^{\sin(3x^2)} \]

\[
\begin{array}{cc}
N & I \\
1 & 2.75039 \\
10 & 1.9893 \\
100 & 1.79139 \\
1000 & 1.75146 \\
10000 & 1.77313 \\
100000 & 1.77862 \\
\end{array}
\]

Monte Carlo Integration

\[
V\{f(X)\} = E\{f(X)^2\} - [E\{f(X)\}]^2
\]
\[
= V \left\{ \frac{1}{N} \sum_{i=1}^{N} f(x_i) \right\}
\]
\[
= \frac{1}{N^2} \sum_{i=1}^{N} V\{f(x_i)\}
\]
\[
= \frac{1}{N} V\{f(x_i)\}
\]

Monte Carlo Convergence

Convergence: \( \sigma \propto \frac{1}{\sqrt{N}} \)

- Slow :-(
- Independent of the dimension

Monte Carlo Integration

Improvements:
- Stratified sampling
- Importance sampling

Stratified Sampling

\[ f(x) = e^{\sin(3x^2)} \]

\[
\begin{array}{cc}
N & I \\
1 & 2.70457 \\
10 & 1.72858 \\
100 & 1.77925 \\
1000 & 1.77606 \\
10000 & 1.77610 \\
100000 & 1.77610 \\
\end{array}
\]

Convergence: \( \sigma \propto \frac{1}{\sqrt{N}} \)
A Visual Break

Monte Carlo Sampling
- Rejection sampling
- PDF based sampling

Rejection Sampling

Sampling a continuous distribution
- Cumulative probability distribution function
  \[ P(x) = \Pr(X < x) \]
- Construction of samples
  - Solve for \( X = P^1(U) \)
- Must know:
  1. The integral of \( p(x) \)
  2. The inverse function \( P^{-1}(x) \)

PDF Based Sampling

Monte Carlo Sampling
- Integrate over domain
  - Disc
  - Triangle
  - Hemisphere
- Sample using ray tracing

Rejection Sampling a Disc

```
do {
    \( X = 1 - 2 \cdot U_1 \)
    \( Y = 1 - 2 \cdot U_2 \)
    while (X^2 + Y^2 > 1)
}
```

May be used to pick random 2D directions
Circle techniques may also be applied to the sphere
Direct Sampling of a Disc

\[ A = \frac{1}{2} \int_0^1 2\pi r^2 \, dr \, d\theta = \frac{1}{2} \int_0^1 r^2 \, dr \, \int_0^{2\pi} \, d\theta = \left[ \frac{r^3}{3} \right]_0^1 \pi = \pi \]

\[ p(r, \theta) \, dr \, d\theta = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi} \]

\[ p(\theta) = \frac{1}{2\pi}, \quad \theta = 2\pi U_1 \]

\[ P(\theta) = \frac{1}{2\pi} \theta = U_2 \]

\[ p(r) = 2r, \quad P(r) = r^2 \]

\[ r = \sqrt{U_2} \]

Sampling of a Disc

Wrong ≠ Equi-Areal

\[ \theta = 2\pi U_1 \]

\[ r = U_2 \]

Right = Equi-Areal

\[ \theta = 2\pi U_1 \]

\[ r = \sqrt{U_2} \]

Sampling a Triangle

\[ u \geq 0 \]

\[ v \geq 0 \]

\[ u + v \leq 1 \]

\[ u + v = 1 \]

\[ A = \int_0^1 \int_0^{1-u} dv \, du = \int_0^1 (1-u) \, du = -\frac{(1-u)^2}{2} \bigg|_0^1 = \frac{1}{2} \]

\[ p(u, v) = 2 \]

Sampling a Triangle

Here \( u \) and \( v \) are not independent!

\( p(u, v) = 2 \)

Conditional probability

\[ p(u) = \int p(u, v) \, dv = 2(1-u) \]

\[ P(u_u) = \int_0^{1-u} (1-u) \, du = (1-u)^2 \]

\[ p(u | v) = \frac{1}{(1-u)} \]

\[ P(v | u_u) = \int_0^{1-u} \frac{1}{(1-u)} \, dv = \frac{v}{(1-u)} \]

Diffuse Lighting

\[ L(x, \vec{\omega}) = \frac{R_d}{\pi} \int_{2\pi} L_o(x, \vec{\omega}') (\vec{n} \cdot \vec{\omega}') \, d\vec{\omega}' \]

- Accounts for all direct light on a diffuse surface
- Use Monte Carlo integration

Sphere Light

Naive rejection sampling
Sphere Light

Explicit integration - sampling the light

Glossy Phong

\[ p(\theta, \phi) = \frac{n + 1}{2\pi} \cos^n \theta \]

\[ P(\theta, \phi) = \int_0^\theta \int_0^{2\pi} p(\theta', \phi') \sin \theta' \, d\theta' \, d\phi' \]

\( (\theta, \phi) = (\arccos((1 - u)1/(n+1)), 2\pi v) \)

Glossy Spheres

1 sample

Glossy Spheres

256 samples

Next time

- Global illumination
  - The rendering equation
  - Light transport
  - More Monte Carlo...
  - Finite Element Radiosity