CSE168
Computer Graphics II, Rendering

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Last time

- Part I: Implementing a basic ray tracer
- Part 2: Physics of light transport
- Part 3: Advanced topics
Last time

- Solid angle
- Radiometry
- Reflection integral
- Monte Carlo integration: basic concepts
Solid angle

- Area on the unit sphere that is spanned by a set of directions
- Unit: steradian $sr$
- Directions usually denoted by $\omega$
Differential solid angle

Differential solid angle: the differential area on the unit sphere around direction $\omega$

$$d\omega = \sin \theta d\theta d\phi$$
Differential solid angle

Differential solid angle spanned by differential area $dA$ at distance $r$ and angle $\psi$

$$d\omega = \frac{dA \cos \psi}{r^2}$$
Radiometry

Measuring flow of light energy, “counting photons”
Radiometric quantities

- Power, or radiant flux: energy per time
  \[ \Phi(\omega, x) = \frac{dQ(\omega, x)}{dt}, \quad [W = J \cdot s^{-1}] \]

- Radiance: power per solid angle per area
  \[ L(\omega, x) = \frac{d^2\Phi(\omega, x)}{d\omega dA\perp}, \quad [W \cdot sr^{-1} \cdot m^{-2}] \]
Radiometric quantities

- Irradiance: power per area
  \[ E(x, n) = \frac{d\Phi(x)}{dA}, \text{[W} \cdot m^{-2}] \]

- Intensity: power per solid angle
  \[ I(\omega) = \frac{d\Phi(\omega)}{d\omega}, \text{[W} \cdot sr^{-1}] \]
Irradiance

Integrating radiance over the hemisphere

\[ E(x, n) = \int_{\mathcal{H}^2(n)} L_i(x, \omega) \cos \theta \, d\omega \]
The BRDF (revisited)

Bidirectional reflectance distribution function

- The BRDF is the fraction of differential reflected radiance over differential irradiance

\[
f(x, \omega_o, \omega_i) = \frac{dL_o(x, \omega_o)}{dE(x, \omega_i)} = \frac{dL_o(x, \omega_o)}{L_i(x, \omega_i) \cos \theta_i d\omega_i}, \quad [sr^{-1}]\]
Reflection equation

- Outgoing radiance due to incident illumination from all directions

\[ L_o(x, \omega_o) = \int_{\mathcal{H}^2(n)} f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos \theta_i \, d\omega_i \]
Numerical computation

\[ L_o(x, \omega_o) = \int_{H^2(n)} f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos \theta_i \, d\omega_i \]
Monte Carlo integration

- We want to compute
  \[ \int_{a}^{b} f(x) \, dx \]
- The Monte Carlo estimator is
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \]
  where \( X_i \) are random variables with distribution \( p(x) \)
- Works also for multidimensional integrals
Monte Carlo integration

• Unbiased estimator

\[ E[F_N] = \int_a^b f(x) \, dx \]

• Expected error converges with \( O(N^{-1/2}) \)

• To get half the error, we need to quadruple the number of samples

• Slow convergence ...
Today

- Surface form of reflection equation
- Sampling random variables
- Area light sources
- Improving the efficiency of the Monte Carlo estimator
Surface form of reflection equation

• So far: directional form of reflection equation

\[ L_0(x, \omega_o) = \int_{\mathbb{H}^2(n)} f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos \theta_i d\omega_i \]

• Surface form: instead of integrating over all directions \( \omega_i \) on the hemisphere, integrate over all visible surfaces
Surface form of reflection equation

- Solid angle spanned by differential surface element $dA$ at location $x'$ with normal $n$

\[ d\omega = \frac{dA \cos \psi}{\|x' - x\|^2} \]

where

\[ \cos \psi = \frac{n \cdot (x - x')}{\|x' - x\|} \]
Surface form of reflection equation

- Reflection integral over all visible surfaces

\[ L_o(x, \omega_o) = \int_{\text{all } x' \text{ visible to } x} \frac{f(x, \omega_o, \omega_i) L_i(x, \omega_i) \cos \theta_i \cos \psi}{\|x' - x\|^2} \, dA \]

where \( \omega_i = \frac{x' - x}{\|x' - x\|} \)
Surface form of reflection equation

- Reflection integral over all surfaces using visibility function

\[ L_o(x, \omega_o) = \int_{\text{all } x'} \frac{f(x, \omega_o, \omega_i) L_i(x, \omega_i) v(x, x') \cos \theta_i \cos \psi}{||x' - x||^2} dA \]

where the visibility function is

\[ v(x, x') = \begin{cases} 
1 & \text{if } x \text{ and } x' \text{ are mutually visible} \\
0 & \text{otherwise.} 
\end{cases} \]
Monte Carlo estimate

- **Directional form**

\[
L_o(x, \omega_o) \approx \sum_j \frac{f(x, \omega_o, \omega_{i,j}) L_i(x, \omega_{i,j}) \cos \theta_i}{p(\omega_j)}
\]

- **Surface form**

\[
L_o(x, \omega_o) \approx \sum_j \frac{f(x, \omega_o, \omega_{i,j}) L_i(x, \omega_{i,j}) v(x, x'_j) \cos \theta_i \cos \psi}{\|x'_j - x\|^2 p(x'_j)}
\]
Monte Carlo estimate

- How to define suitable distributions $p$
- How to sample random variables $\omega_j, x'_j$
- Lots of technical detail
- Today, we focus on surface form and sampling $x'_j$
- First, some theory...
Example: uniform sampling of a disk

- Uniform probability density on a unit disk
  \[ p(x, y) = \begin{cases} 
  1/\pi & x^2 + y^2 < 1 \\
  0 & \text{otherwise.} 
\end{cases} \]

- Goal: draw samples \( X_i, Y_i \) that are distributed with
  \[ (X_i, Y_i) \sim p(x, y) \]
Example: uniform sampling of a disk

- Problem: pseudo random number generators allow us only to draw samples $\xi_i$ from the canonical distribution

$$\xi_i \sim p(\xi) = \begin{cases} 1 & \xi \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$
Example: uniform sampling of a disk

Rejection sampling

- Draw samples $\xi_1, \xi_2$
- Reject if $\xi_1^2 + \xi_2^2 > 1$
- Drawback: inefficient

- rejected
- accepted
Example: uniform sampling of a disk

- Idea: transform samples to polar coordinates

\[ r = \xi_1 \quad \theta = 2\pi \xi_2 \]

- Problem?
Sampling arbitrary distributions

The inversion method

- Discrete example: 4 events with probabilities $p_1 \ p_2 \ p_3 \ p_4$
Sampling arbitrary distributions

• “Draw samples”, ”sampling” the distribution: generate events with appropriate probabilities
Sampling arbitrary distributions

The inversion method, discrete example

- Compute the discrete CDF

Discrete CDF

1
Sampling arbitrary distributions

The inversion method, discrete example

- Sample canonic random variable $\xi$
- Generate event intersected by $\xi$

```
 1
\xi
```

```
  p_1  p_1  p_1  p_1
  p_2  p_2  p_2
    p_3  p_3
      p_4
```

Discrete CDF
Sampling arbitrary distributions

The inversion method, continuous case

Goal: sample from arbitrary distribution $p(x)$

1. Compute the CDF $P(x) = \int_0^x p(x') dx'$
2. Compute its inverse $P^{-1}(x)$
3. Obtain a uniformly distributed random number $\xi$
4. Compute $X_i = P^{-1}(\xi)$
Sampling 2D distributions

- Draw samples \((X, Y)\) from 2D distribution \(p(x, y)\)

- If \(p(x, y)\) is separable, i.e., \(p(x, y) = p_x(x)p_y(y)\) we can independently sample \(p_x(x), p_y(y)\)
Sampling 2D distributions

• Otherwise, first compute the marginal density function

\[ p(x) = \int p(x, y) dy \]

• Then, find the conditional density

\[ p(y|x) = \frac{p(x, y)}{p(x)} \]

• Procedure: first sample \( X_i \sim p(x) \), then \( Y_i \sim p(y|x) \)
Transforming between distributions

- Given $X_i \sim p_x(x)$
- What is the distribution of $Y_i = y(X_i)$?
- Note that $y(x)$ must be one-to-one
- This implies

$$\Pr\{Y \leq y(x)\} = \Pr\{X \leq x\}$$

$$P_y(y) = P_y(y(x)) = P_x(x)$$
Transforming between distributions

- Differentiating $P_y(y(x)) = P_x(x)$ using the chain rule yields

$$p_y(y) \frac{dy}{dx} = p_x(x)$$

$$p_y(y) = \left(\frac{dy}{dx}\right)^{-1} p_x(x)$$
Transforming between distributions

• With n-dimensional random variable $X$
• Bijective mapping $Y = T(X)$
• Densities
  $$p_y(y) = p_y(T(x)) = \frac{1}{|J_T(x)|} p_x(x)$$
  where $|J_T| = |\text{det}(J)|$
• The Jacobi matrix is
  $$J_T = \begin{bmatrix}
  \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n}
\end{bmatrix}$$
Transforming between distributions

- The same mechanism as for change of variables in multi-dimensional integration
Example: uniform sampling of a disk

- Desired distribution

\[ p(x, y) = \begin{cases} 
\frac{1}{\pi} & x^2 + y^2 < 1 \\
0 & \text{otherwise.} 
\end{cases} \]

- We will sample polar coordinates \( r, \theta \), where \( x = r \cos \theta, y = r \sin \theta \)

- Jacobi matrix

\[
J_T = \begin{bmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}
\end{bmatrix} = \begin{bmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{bmatrix}
\]
Example: uniform sampling of a disk

- **Determinant is** \( r(\cos^2 \theta + \sin^2 \theta) = r \)
- Therefore
  
  \[
p(x, y) = \frac{p(r, \theta)}{r}  
  \]
  
  \[
p(r, \theta) = rp(x, y) = \frac{r}{\pi}  
  \]

- **Marginal and conditional distributions**
  
  \[
p(r) = \int_{0}^{2\pi} p(r, \theta) \, d\theta = 2r  
  \]
  
  \[
p(\theta|r) = \frac{p(r, \theta)}{p(r)} = \frac{1}{2\pi}  
  \]
Example: uniform sampling a disk

• Sampling from $p(r)$ and $p(\theta | r)$ by integrating the PDFs and inverting

\[
\xi_1 = P(r) = r^2 \Rightarrow r = P^{-1}(\xi_1) = \sqrt{\xi_1}
\]

\[
\xi_2 = P(\theta) = \theta/(2\pi) \rightarrow \theta = P^{-1}(\xi_2) = 2\pi \xi_2
\]
Example: uniform sampling a disk

WRONG ≠ Equi-Areal

\[ r = \xi_1 \]
\[ \theta = 2\pi \xi_2 \]

RIGHT = Equi-Areal

\[ r = \sqrt{\xi_1} \]
\[ \theta = 2\pi \xi_2 \]
Recipe

1. Express the desired distribution in a convenient coordinate system. **Note:** requires computing the determinant of the Jacobian.

2. Compute marginal and conditional 1D PDFs.

3. Sample 1D PDFs using the inversion method.
Other useful examples

Sampling distributions on

• Rectangles
• Triangles
• Spheres
• Hemispheres
Area lights

Recall: the surface form of the reflection equation

\[ L_o(x, \omega_o) \approx \sum_j \frac{f(x, \omega_o, \omega_{i,j}) L_i(x, \omega_{i,j}) v(x, x'_j) \cos \theta_i \cos \psi}{\|x'_j - x\|^2 p(x'_j)} \]
Area lights

- Assume uniform sampling of area light
  \[ p(x'_j) = \frac{1}{A} \]
- Assume isotropic light source
  \[ L_i(x, \omega_{i,j}) \equiv L_i = \frac{\Phi}{(2\pi A)} \]
Area lights

shade_area_light( x, omega_o, n) {  
  for( j=1; j<N; j++) {  
    choose random point xp with normal np  
      on light source  
    d = xp - x  
    omega_i = d/length(d)  
    if( light is visible along ray x+td ) {  
      return f( omega_i, omega_o ) *  
        L_i * dot( n,d ) *  
        ( -np, d ) * A / length( d )^4  
    }  
  }  
}
Area lights

one shadow ray

lots of shadow rays
Area lights

Directional form of reflection equation

Surface form of reflection equation

Same number of sample

[Pharr]
Area lights

Extensions

• Joint sampling of multiple lights
• Non-isotropic light sources (spot lights)
• Lighting from spherical environment
Lighting from environments

[Will Chang]
Improving efficiency

- Stratified sampling
- Russian roulette
- Importance sampling
Importance sampling

- Recap: Monte Carlo estimator
  \[
  F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}
  \]
- Imagine we had
  \[
  p(x) \propto f(x), \text{ or } p(x) = cf(x)
  \]
- Normalization forces
  \[
  c = 1 / \int f(x) dx
  \]
- Now
  \[
  \frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x) dx
  \]
Importance sampling

- Directional form of reflection equation

\[ L_o(x, \omega_o) \approx \sum_j \frac{f(x, \omega_o, \omega_{i,j})L_i(x, \omega_{i,j}) \cos \theta_i}{p(\omega_j)} \]

- Choose \( p(\omega_j) \)
  - According to BRDF
  - According to incoming light
  - Multiple importance sampling: optimal trade-off between both
Importance sampling

Sampling more the light

Sampling more the BRDF

[Veach, Guibas]
Multiple importance sampling

Naïve sampling

Optimal sampling

[Veach, Guibas]
Next time

• Global illumination
• Wojciech!